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Pierre Anglès

Conformal Groups in Geometry and Spin Structures

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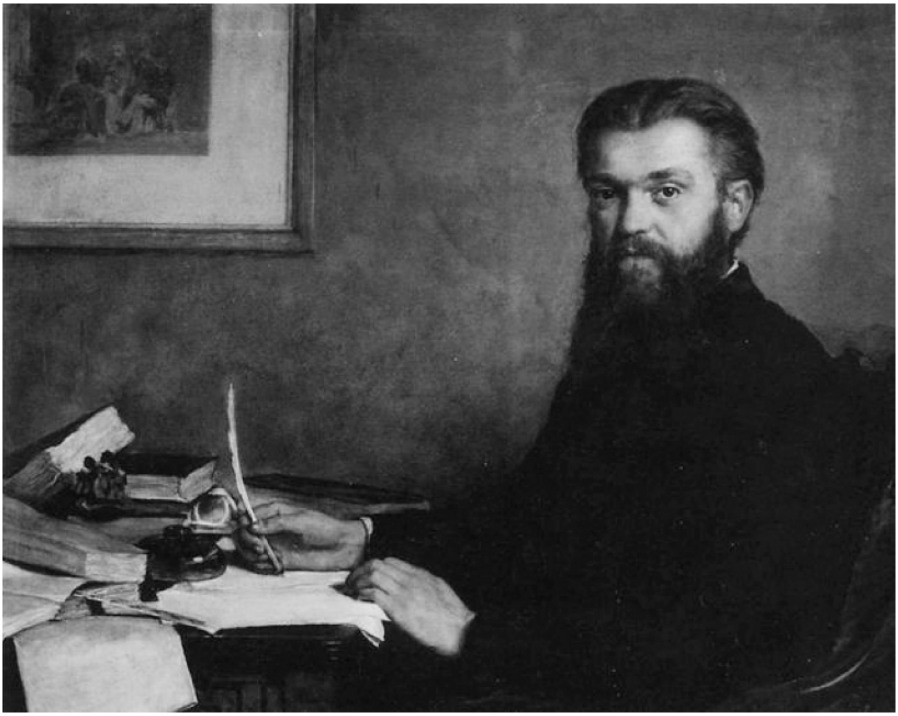
To the memory of my grandparents and my father, Camille;
to my mother, Juliette, my wife, Claudie,
my children, Fabrice, Catherine and Magali,
my grand-daughters Noémie, Elise and Jeanne
and to the memory of my friend Pertti Lounesto.



William K. Clifford (1845–1879), Mathematician and Philosopher. Portrait by John Collier (by kind permission of the Royal Society).

"The Angel of Geometry and the Devil of Algebra fight for the soul of any mathematical being."

Attributed to Hermann Weyl
(Communicated by René Deheuvels himself
according to a private conversation with H. Weyl)

"C'est l'étude du groupe des rotations (à trois dimensions) qui conduisit Hamilton à la découverte des quaternions; cette découverte est généralisée par W. Clifford qui, en 1876, introduit les algèbres qui portent son nom, et prouve que ce sont des produits tensoriels d'algèbres de quaternions ou d'algèbres de quaternions et d'une extension quadratique.

Retrouvées quatre ans plus tard par Lipschitz qui les utilisa pour donner une représentation paramétrique des transformations à n variables ... ces algèbres et la notion de '*spineur*' qui en dérive, devaient aussi connaıtre une grande vogue à l'époque moderne en vertu de leur utilisation dans les théories quantiques."

Nicolas Bourbaki
Eléments d'histoire des Mathématiques
Hermann, 1969, p. 173.

Foreword

It is not very often the case that a treatise and textbook is called to become a standard reference and text on a subject. Generally a comprehensive treatment on a subject is devoted to the specialist and a didactical textbook is a newer version of a series of guiding monographs. This book by Pierre Anglès is all these things in one: a good reference on the subject of Clifford algebras and conformal groups and the subjacent spin structures, a textbook where students and even specialists of any one of the subjects can learn the full matter, and a bridge between the basic approach of Grassmann and Clifford of finding a linear form that corresponds to a given quadratic form and all the structures which can be built from those algebras and in particular the pseudounitary conformal spin structures.

The numerous references, starting in the foreword itself and within each chapter supply the necessary connection to the state of the art of the subject as viewed by numerous other authors and the creative contributions of Professor Anglès himself. A fresh approach to the subject is found anyway and this characteristic is the basis for this book to become, as we said, a standard text and reference.

Besides the rigorous algebraic approach a consistent geometrical point of view, in the genealogy of Wessel, Argand, Grassmann, Hamilton, Clifford, etc. and of Cartan and Chevalley is found throughout the book. In fact it would be desirable that this transparency of presentation would be continued one day, by Professor Anglès, in the field of mathematical physics and perhaps even in theoretical physics where a clear connection between algebra, geometry and spin structures with physical theoretical structures are always welcome. The same applies to the possibility of extending, in the future, the numerous present exercises, which are a guidance for the study of the subject, to applications in other branches of mathematics and theoretical physics.

We finally want to stress that the effort of the author to clearly present the development from Clifford algebras through conformal real pseudo-euclidean geometry, pseudounitary conformal spin structures and more advanced applications has resulted in fact in abundant new concepts and material.

Jaime Keller
University of Mexico

Foreword

During the second part of the 19th century a large number of important algebraic structures were discovered. Among them, quaternions by Hamilton and the exterior calculus or multilinear algebra by Sylvester, are by now part of standard textbooks in algebra or geometry. Since its discovery by W.K. Clifford, the Clifford algebras, a sort of mixture of the above-mentioned structures, very quickly emerged as a fundamental idea. In the same way as quaternions extend the dream of complex numbers to dimensions three and four, the Clifford idea of adding a formal square root of a quadratic quantity works marvellously in any dimension. Very soon it was the Clifford construction is correlated to classical geometry. This relationship is now clearly explained mostly in terms of the spin group, which is the group counterpart of the Clifford algebra.

Physicists also quickly recognized the importance of the spin group and its spin representation, both in Euclidean and Minkowski signatures. The word “spin” is almost a genetic term of the quantum theory, and of the physics of elementary particles. More recently, the development of the idea of supersymmetry shows that vitality and modernity are in perfect accord with the structures introduced by Clifford. Clifford, spinors, and Poincaré algebras are at the heart of this fascinating idea.

The book of P. Anglès intertwines both the algebraic and geometric viewpoints. The first half of this book is algebraic in nature, and the second half emphasizes the differential-geometric side. Many books are devoted totally or in part to the Clifford algebras with an algebraic viewpoint. Then the results are often corollaries of the structure theorems of semisimple algebras, the Wedderburn theory. The point of view of the present book is more pragmatic. The whole theory is explained in a concise but very explicit manner, referring to standard textbooks for the general tools. A whole battery of exercises helps the reader to master the intricacies of the numerous structural results offered to the reader.

In the geometric chapters, dealing with vector bundles over manifolds with extra structures, spinorial, conformal, and many others, the same pedagogic treatment is proposed. I am convinced that this is a good point of view. It makes the presentation of these rather subtle structures particularly clear and sometimes exciting. Numerous exercises complete the text in many directions, adding supplementary material. All this makes the book essentially self-contained.

The book of P. Anglès is neither a textbook of algebra, nor a treatise on differential geometry, but a book of old and new developments concerning the puzzle around Clifford's ideas. I recommend this book to any student or researcher in mathematics or physics who wants to master this exciting subject.

José Bertin
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Preface

Since 1910, has been well known not only that Maxwell's equations are invariant for the 10-dimensional Poincaré group (or inhomogeneous Lorentz group), but that the maximal group of invariance is the 15-dimensional conformal group $C(1, 3)$ of the classical Minkowski standard space $E_{1,3}$, which is the smallest semisimple group containing the Poincaré group. We recall that the Poincaré group is the semidirect product of the (homogeneous) Lorentz group by the group of the translations: $\mathcal{T}(E_{1,3})$. Many attempts have been made to build up a new theory of relativity, to find a cosmology, or to reveal classifications of elementary particles from the study of the conformal groups. The twistor theory of Roger Penrose is such an example, and its success is ever increasing.[1]

The structure of the classical pseudoorthogonal group $\mathrm{SO}^+(2, 4)$ had been already studied by Elie Cartan, who had shown[2] the identity of the Lie algebras of $C(1, 3)$ and $SU(2, 2)$. Physicists who need conformal pseudoorthogonal groups use only their Lie algebras. The fundamental idea of the theory of Penrose is that $SU(2, 2)$ is a fourfold covering of the connected component of $C(1, 3)$. A twistor is nothing but a vector of the complex space $\mathbf{C}^4$ provided with the standard pseudo-hermitian form of signature (2, 2), and the submanifold of the Grassmannian of complex spaces of $\mathbf{C}^4$ constituted by totally isotropic planes is identical to the conformal compactified space of the Minkowski space $E_{1,3}$. *We can associate canonically with each n-dimensional quadratic space (E, q) an associative unitary algebra: its Clifford algebra $\mathbf{C}(E, q)$.*

Historically, the notion of Clifford algebras naturally appeared in many different ways. Its destiny is closely joined to the development of generalized complex numbers and the success of the theory of quadratic forms.

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[1] R. Penrose, Twistor algebra, *J. of Math. Physics*, vol. 8, no. 2, 1967; Ward and Wells, *Twistor Geometry and Field Theory*, Cambridge University Press; H. Blaine Lawson and M. L. Michelson, *Spin Geometry*, P. U. Press, 1989; N. Woodhouse, *Geometry Quantization*, Clarendon Press, 1980, etc.

[2] Elie Cartan, *Annales de l'E.N.S.*, 31, 1974, pp. 263–365.

The story of complex imaginary numbers starts in the sixteenth century when Italian mathematicians Girolamo Cardano (1501–1576), Raphaele Bombelli (born in 1530, whose algebra was published in 1572), and Niccolo Fontana, called “Tartaglia”—which means stammerer—realizing that a negative real number cannot have a square root, began to use a symbol for its representation. *Thus came into the world the symbol i, such that* $i^2 = -1$*, a very mathematical oxymoron, the success of which is well known.*[3]

The introduction of generalized complex numbers of order more than 2 is not quite linked to the solution of equations of order two with real coefficients. Their destiny is closely joined to the attempts is made by Gaspar Wessel in 1797 and by J. R. Argand, J. F. Français, F. G. Servois from 1814 to 1815 in order to extend the geometrical theory of imaginary numbers of the plane to the usual space.

We recall that, starting from the classical field **R** of real numbers, we can define the three following generalized numbers of order two:[4]

Classical complex numbers (or elliptic numbers): $a + ib$, $a, b \in \mathbf{R}$, $i^2 = -1$;

Dual numbers (or parabolic numbers): $a + \mathcal{E}b$, $a, b \in \mathbf{R}$, $\mathcal{E}^2 = 0$;

Double numbers (or hyperbolic numbers): $a + eb$, $a, b \in \mathbf{R}$, $e^2 = 1$.

W. R. Hamilton,[5] professor of astronomy in the University of Dublin, was the first to introduce in 1842 a system of numbers of order $2^2 = 4$, with a noncommutative multiplicative law: the sfield **H**. The study of the group of rotations in the classical 3-dimensional space led W. R. Hamilton to his discovery.

Dual and double numbers were studied by two mathematicians: Eugène Study (1862–1930) and William Kingdom Clifford (1845–1879). The applications of these new objects belong to the increasing success of non-Euclidean geometries. Moreover, W. K. Clifford introduced in 1876 the algebras that are called Clifford algebras in a lecture published in 1882, after his death. The work of W. K. Clifford was completed by that of R. O. Lipschitz in 1886. As for the term “spinor,” its destiny probably begins with Leonhard Euler (1770) and Olinde Rodrigues (1840).[6]

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[3] The word was first used by the French mathematician and philosopher René Descartes (*Géométrie*, Leyde, 1637, livre 3), and R. Bombelli (*Algebra*, Bologna, Italy, 1572, p. 172) used the expression “piu di meno” for $\sqrt{-1}$ and “meno di meno” for $-\sqrt{-1}$. We recall that an oxymoron is a rhetorical figure that joins two opposite words such as: a dark clearness, a deafening silence.

[4] W. K. Clifford, Applications of Grassmann’s extensive algebra, *American Journal of Mathematics*, 1 (1878), pp. 350–358; and W. K. Clifford, *Mathematical Papers*, London, Macmillan, 1882.

[5] W. R. Hamilton considered the set of numbers z, $z = a + ib + jc + kd$, where a, b, c, d belong to **R**, with the usual addition and the following multiplicative table for the “units” i, j, k: $i^2 = j^2 = k^2 = -1$, $ij = k$, $jk = i$, $ki = j$, $kj = -i$, $ji = -k$, $ik = -j$.

[6] Cf. E. Cartan, Nombres Complexes, Exposé d’après l’article allemand de E. Study, Bonn, *Œuvres Complètes*. Partie II Volume 1, pp. 107–408, Gauthier Villars, Paris 1953; and Paolo Budinich and Andrzej Trautman, An introduction to the spinorial chessboard, *J.G.P.*, no. 3, 1987, pp. 361–390, and *The Spinorial Chessboard*, Springer-Verlag, 1968.

According to B. L. Van der Waerden,[7] the name "spinor" is due to Paul Ehrenfest. The discovery of quaternions by William Rowan Hamilton[8] led to a simple "spinorial" representation of rotations. If $q = ia + jb + kc$ is a "pure" quaternion and u is a unit quaternion, then $q \to uqu^{-1}$ is a rotation and every rotation can be so obtained. *The way to spinors initiated by L. Euler, completed by W. K. Clifford and R. O. Lipschitz,*[9] *is based on the fundamental idea of taking the square root of a quadratic form.*

Among the various ways that lead to Clifford algebras, the most spectacular route incontestably appears to be the solution given by P. A. M. Dirac[10] to the problem of the relativistic equation of the electron, when he sought and linearized the Klein–Gordon operator, which is the restricted relativistic form of the equation of Schrödinger:

$$(\Box - m^2)\psi = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} - m^2\right)\psi = 0, \tag{I}$$

where ψ is a wave function and m a nonnegative real. Physical interpretation of ψ needs to avoid the presence of $\partial^2/\partial t^2$ in (I), and thus led P. A. M. Dirac to writing

$$\begin{aligned}\Box - m^2 = &\left(\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y} + \delta\frac{\partial}{\partial z} - m\right)\\ &\times\left(\alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y} + \delta\frac{\partial}{\partial z} + m\right)\end{aligned} \tag{II}$$

as a product of first-order linear operators.

By identifying both members of relation (II), one obtains

$$\alpha^2 = -\beta^2 = -\gamma^2 = -\delta^2 = 1,$$
$$\alpha\beta + \beta\alpha = \alpha\gamma + \gamma\alpha = \cdots = 0.$$

Moreover, a solution can be expected only if the coefficients $\alpha, \beta, \gamma, \delta$ need to be added, multiplied by real numbers, and multiplied between themselves, and, according to (II), belong to a noncommutative algebra. Up to isomorphism, there exists a unique solution obtained by taking for $\alpha, \beta, \gamma, \delta$ complex square matrices of order 4: the Dirac matrices. Mathematically speaking, the problem is a special case of the following

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7 B. L. Van der Waerden, Exclusion principle and spin, in *Theoretical Physics in the Twentieth Century: A Memorial Volume to Wolfgang Pauli*, ed. M. Fierz and V. F. Weisskopf, New York: Interscience, 1960.

8 W. R. Hamilton, *Lectures on Quaternions*, London Edinburgh Dublin Philos. Mag. 25, 1884, p. 36, p. 489, cf. also, W. R. Hamilton, *Elements of Quaternions*, London, 1866, edited by his son W. E. Hamilton, 2nd edition published by Ch. J. Joly 1, London 1899, 2 London, 1901, translated into German by P. Glan, Leipzig, 1882.

9 The algebras considered by Clifford and Lipschitz were generated by n anticommuting "units" e_α with squares equal to -1.

10 P. A. M. Dirac, *Proceedings of the Royal Society*, vol. 117, 1917, p. 610 and vol. 118, 1928, p. 351.

one: *Let E be a space over a field K, endowed with a quadratic form q: how can one express q as the square of a linear form* φ*, i.e., for all* $m \in E$*, how can one express* $q(m)$ *as* $q(m) = (\varphi(m))^2$ *with* φ *belonging to the dual* E^* *of E?* And the special case solved by the physicist Dirac is that of $\mathbf{R}^4$ endowed with the quadratic Lorentz form defined for all $m = (t, x, y, z) \in \mathbf{R}^4$ by $q(m) = t^2 - x^2 - y^2 - z^2$ and the search of a linear form φ defined on $\mathbf{R}^4$, $\varphi(m) = \alpha t + \beta x + \gamma y + \delta z$ such that $q(m) = (\varphi(m))^2$.

The notion of spinor had been formulated by Elie Cartan[11] while he was seeking to determine linear irreducible representations of the proper orthogonal group or of the corresponding Lie algebra. The algebraic presentation of the theory of spinors was first developed in the neutral case by Claude Chevalley.[12] Many other authors such as Albert Crumeyrolle,[13] René Deheuvels,[14] and Pertti Lounesto[15] have taken in interest in such a theory. Besides, the algebraic theory of quadratic forms and Clifford algebras for projective modules of finite type was formulated by Artibano Micali and Orlando Villamayor.[16] The links between Clifford algebras and K-theory have been developed by M. Karoubi.[17] We add that Ichiro Satake[18] used these algebraic tools in an important book. The work of J. P. Bourgignon in the application of Clifford algebras to differential geometry and that of Rod Gover, as well as of the late Thomas Branson must be recalled.

In Clifford analysis, the work initiated by Richard Delanghe and the Belgian school, with F. Brackx and F. Sommen[19] must be emphasized. Guy Laville, Wolfgang Sprössig and John Ryan need also to be recalled together with the late J. Bures.

In addition, the Clifford community knows the work done by Paolo Budinich, Roy Chisholm and William Baylis in mathematical and theoretical physics. David Hestenes cannot be forgotten for his geometric calculus, his fundamental geometric algebra and his part played in many other offshoots of Clifford algebras, together with Jaime Keller and his elegant theory "START," and Waldyr A. Rodrigues Jr. and

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[11] Elie Cartan, *Leçons sur la Théorie des Spineurs I et II*, edition Hermann, Paris, 1937; or *The Theory of Spinors*, Hermann, Paris 1966.

[12] Claude Chevalley, *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954.

[13] A list of publications of the late A. Crumeyrolle is given at the end of the first chapter.

[14] R. Deheuvels published two books: *Formes Quadratiques et Groupes Classiques*, P.U.F., Paris 1991, and *Tenseurs et Spineurs*, P.U.F., Paris 1993.

[15] My friend the late Pertti Lounesto, who was called the Clifford policeman, published a book: *Clifford Algebras and Spinors*, Cambridge University Press, 2nd edition, 2001.

[16] A. Micali and O. Villamayor, Sur les algèbres de Clifford, *Annales Scientifiques de l'Ecole Normale Supérieure*, 4° serie, tome 1, 1968, pp. 271–304.

[17] M. Karoubi, Algèbres de Clifford et K-theorie, *Annales de l'E.N.S.*, 4° serie, tome 1, 1968, pp. 161–270.

[18] I. Satake, *Algebraic Structures of Symmetric Domains*, Iwanani Shoten, Publishers and Princeton University Press, 1980.

[19] F. Brackx, R. Delanghe, F. Sommen, Clifford Analysis, Pitman Publ., Boston-London-Melbourne, 1982.

Y. Friedmann for their important work in fundamental physics. Rafal Abłamowicz has studied many applications of Clifford algebras such as in computing science and took also an interest with Z. Oziewicz and J. Rzewuski in the study of twistors. Arkadiusz Jadczyk came to the study of Clifford algebras after that of many other subjects. He is an innovator for the links between Clifford algebra and quantum jumps.

The following self-contained book can be used either by undergraduates or by researchers in mathematics or physics. Before each chapter a brief introduction presents the aims and the material to be developed. Chapter 1 is also a chapter of reference. Each chapter presents its own exercises with its own bibliography.

Pierre Anglès
Institut de Mathématiques de Toulouse

Overview

The first chapter is devoted to the presentation of the necessary algebraic tools for the study of Clifford algebras and to a systematic study of different structures given to the spaces of spinors for even Clifford algebras $C^{+}_{r,s}$ of quadratic regular standard spaces $E_{r,s}$ and of the corresponding embeddings of associated spin groups and projective quadrics. Many exercises are proposed.

The second chapter deals with conformal real pseudo-Euclidean geometry. First, we study the classical conformal group of the standard Euclidean plane. Then, we construct covering groups for the general conformal group $C_n(p, q)$ of a standard real space $E_n(p, q)$. We define a natural injective map that sends all the elements of the standard regular space $E_n(p, q)$ into the isotropic cone of $E_{n+2}(p + 1, q + 1)$, in order to obtain an algebraic isomorphism of Lie groups between $C_n(p, q)$ and $PO(p+1, q+1)$. The classical conformal orthogonal flat geometry is then revealed. Explicit matrix characterizations of the elements of $C_n(p, q)$ are given. Then, we define new groups called conformal spinoriality groups. The study of conformal spin structures on Riemannian or pseudo-Riemannian manifolds can now be made. The conformal spinoriality groups previously introduced play an essential part. The links between classical spin structures and conformal spin ones are emphasized. Then we can study Cartan and Ehresmann connections and conformal connections. The study of conformal geodesics is then presented. Generalized conformal connections are then discussed. Vahlen matrices are presented. Many exercises are given.

The third chapter is devoted essentially to the study of pseudounitary conformal spin structures. First, we present pseudounitary conformal structures over a $2n$-dimensional almost complex paracompact manifold V and the corresponding projective quadrics $\tilde{H}_{p,q}$ associated with the standard pseudo-hermitian spaces $\mathbf{H}_{p,q}$. Then, we develop a geometrical presentation of a compactification for pseudo-hermitian standard spaces, in order to construct the pseudounitary conformal group of $\mathbf{H}_{p,q}$, denoted by $CU_n(p, q)$. We study the topology of the projective quadrics $\tilde{H}_{p,q}$ and the "generators" of such projective quadrics.

We define the conformal symplectic group associated with a standard real symplectic space $(\mathbf{R}^{2r}, F)$, denoted by $CSp(2r, \mathbf{R})$, where F is the corresponding symplectic form such that $CU_n(p,q) = CSp(2n,\mathbf{R}) \bigcap C_{2n}(2p,2q)$, with the notation of Chapter 2. The Clifford algebra $Cl^{p,q}$ associated with $\mathbf{H}_{p,q}$ is defined. The corresponding spin group $Spin\,U(p,q)$ and covering groups $RU(p,q)$ and $\triangle U(p,q)$ are given associated with a fundamental diagram. The space S of corresponding spinors is defined and provided with a pseudo-hermitian neutral scalar product. The embeddings of spin groups and corresponding quadrics are revealed. Then, conformal flat pseudounitary geometry is studied. Two fundamental diagrams are given. We introduce and give geometrical characterizations of groups called pseudounitary conformal spinoriality groups. The study of pseudounitary spin structures and conformal pseudounitary spin structures over an almost complex $2n$-dimensional manifold V is now presented. The part played by groups called conformal pseudounitary spinoriality groups is emphasized. The links between pseudounitary spin structures and pseudounitary conformal spin ones are given. Exercises are given.

Instructions to the reader

For convenience, we adopt the following rule: 1.2.2.3.2 Theorem means a theorem of Chapter 1, Part 2 Section 2.3.2. At the end of each chapter, we present some references. If we need some reference on a particular page, it will be mentioned by a footnote such as, for example, S. Helgason, Differential geometry and symmetric spaces, op. cit., p. 120. The Lie algebra of a Lie group G will be denoted by $\mathfrak{g}$ or $\mathcal{G}$ or $Lie(G)$ or $\mathcal{L}(G)$. The derivative at x of a map f will be denoted either by $(df)_x$ or by $d_x f$. Sometimes the notation D for d will also be used. By a curve, or path, we shall always mean a curve, or path of at least class C^1. In Chapter 3 $(\mathtt{Sym})_e$ (resp. $(\mathtt{Sym})_{et}$) is sometimes denoted also as $(\mathtt{Sym})_s$ (resp. $(\mathtt{Sym})_{st}$).

Acknowledgments

The author wants to present his warmest thanks to Jaime Keller (University of Mexico) and Wolfgang Sprößig (Technical University of Freiberg) for their reviews of the book and their comments and especially to Jaime Keller for having accepted to prepare a foreword. He wants also to thank José Bertin (University Joseph Fourier of Grenoble, France) especially for his foreword and his critical reading of the manuscript.

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He wants also to present Fernanda Viola with particular thanks for her initial typing of the manuscript.

He wants also to express his thanks to the Advisory Committee of the International Conference(s) on Clifford algebras and their Applications, and specially to J. S. R. Chisholm and W. Baylis, for agreeing to host the seventh International conference in Toulouse May 19–29, 2005, at the University Paul Sabatier and to entrust to him the care of the organization.

Last, but not least, he wants to thank his wife Claudie for her constant support.

Contents

1

Classic Groups: Clifford Algebras, Projective Quadrics, and Spin Groups

The first chapter is a presentation of the necessary algebraic tools. We recall the general classical results concerning general linear groups, unitary groups, symplectic groups and their corresponding Lie algebras, and the same for classical groups over noncommutative fields.

A review of elementary properties of quaternion algebras leads to the study of Clifford algebras, the presentation of the main results concerning such algebras, and the introduction of corresponding spinors and spinor groups (or briefly spin groups) and spin representations. Then, systematically, we study the different structures given to the spaces of the spinors for even Clifford algebras $C^{+}_{r,s}$ of the quadratic standard space $E_{r,s}$, the embeddings of corresponding spin groups $\mathrm{Spin}(r, s)$ and of real projective associated quadrics $\tilde{Q}(E_{r,s})$.

1.1 Classical Groups

A Summary of Classical Results

Hermann Weyl[1] introduced the term "classical group" for summarizing the following groups: linear groups, orthogonal, unitary and symplectic groups. We recall the classical and necessary definitions and results.

1.1.1 General Linear Groups[2]

(a) Let E be a linear space of finite *dimension* n over a field K and let $GL(E)$ denote the set of all linear mappings from E *onto* E (we recall that since E is n-dimensional, these mappings all are bijections). $GL(E)$ is a group under the composition of mappings,

[1] Hermann Weyl, *The Classical Groups*, Princeton University Press, 1936.

[2] See the remarkable book by J. Dieudonné, *La géométrie des Groupes Classiques*, Springer-Verlag, Berlin, 1971, third édition. See also the Encyclopedic Dictionary of Mathematics, edited by Shôkichi Iyanaga and Yukiyosi Kuwada, Cambridge, MA, MIT Press, 1977.

called the general linear group (or full linear group) on E. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of E over K and let $[A_i^j]$ be the matrix associated with the element $f \in GL(E)$ such that

$$f(e_i) = \sum_{j=1}^{n} A_i^j e_j.$$

Then the mapping $f \to [A_i^j]$ *determines* an isomorphism *from* $GL(E)$ *onto the multiplicative group* $GL(n, K)$ *of all square invertible* matrices of degree n, with coefficients in K. *The group* $GL(E)$ is often identified with $GL(n, K)$. $GL(n, K)$ *is called the general linear group of degree n over* K. *The mapping* $u \to \det u$ (where $\det u$ denotes the determinant of u) determines a homomorphism from $GL(E)$ *onto* the multiplicative group $K^* = K - \{0\}$. *The kernel of this homomorphism is a normal subgroup of* $GL(E)$, *denoted by* $SL(E)$ and called the special linear group or unimodular group.

In the same way, $SL(n, K) = \{f \in GL(n, K), \det f = 1\}$ is called the special linear group of degree n over K. The center z of $GL(n, K)$ is identical with the set of all scalar matrices λI, $\lambda \in K^*$, and the center of $SL(n, K)$ is a finite group, namely $\{\lambda I, \lambda \in K^* \text{ and } \lambda^n = 1\}$.

(b) Let us introduce $P(E)$, the projective classical $(n-1)$-dimensional space associated with the n-dimensional K-linear space E (we recall that $P(E)$ can be viewed as the set of all 1-dimensional linear subspaces of E). The projective general linear group on $P(E)$ denoted by $PGL(E)$ is the group of all so-called projective transformations on $P(E)$, that is, $PGL(E) = GL(E)/z$. When $E = K^{n+1}$[3], such a group is denoted by $PGL_n(K)$ or $PGL(n, k)$. $PGL(E) = GL(E)/z$, where z is the center of $GL(E)$ and $PGL(n, k) = GL(n, K)/z$, is called the projective general linear group of degree n. Similarly, $PSL(n, K) = SL(n, K)/z_0$, the quotient group of $SL(n, K)$ by its center, z_0 is called the projective special linear group of degree n.

If the ground field K is either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers, all these groups are respectively Lie groups or complex Lie groups. Thus, $SL(n, \mathbf{C})$ is a simply connected simple and semisimple complex Lie group of type A_{n-1},[4] and $PSL(n, \mathbf{C})$ is the adjoint group of the complex simple algebra of type

[3] We recall that when $E = K^{n+1}$, the projective associated space is also denoted by KP^n.

[4] We recall that the different structures of a compact connected simple Lie group are classically denoted by $A_l (l \geq 1)$, $B_l (l \geq 2)$, $C_l (l \geq 3)$, $D_l (l \geq 4)$, G_2, F_4, E_6, E_7, E_8. Each of these symbols represents a class of groups with isomorphic Lie algebras. The first four, the classical structures, possess a linear well-known representative: for A_l, $SU(l+1)$, the unitary unimodular group of $(l+1)$ complex variables; for C_l, $SpU(l)$, often written $Sp(l)$, the unitary group of l quaternionic variables; for B_l, (respectively D_l), $SO(2l+1)$ (respectively $SO(2l)$), the unimodular orthogonal groups. The quotient group of $SU(n)$, $SpU(n)$ (or $Sp(n)$), $SO(2n)$ by their respective centers, which respectively are cyclic with respectively n, 2, 2 elements, are respectively denoted by $PU(n)$, $PSpU(n)$ (or $PSp(n)$), $PSO(2n)$. The groups $SU(n)$ and $SpU(n)$ (or $Sp(n)$) are simply connected, while $SO(n)$ $(n \geq 3)$ possesses a twofold simply connected covering group Spin(n), the center of which is cyclic of order 2, when n is odd, of order 4 when $n = 2m$, with m odd (i.e., $n \equiv 2 \pmod 4$), and isomorphic

A_{n-1}. The group $PSL(n, K)$, $n \geq 2$ (K = **R**, **C**, **H**), is a noncommutative simple group.

1.1.1.1 Unitary Groups[2]

The set denoted by $U(n)$ of all square unitary matrices of degree n with complex elements is a group under multiplication, called the unitary group of degree n. The normal subgroup of $U(n)$ consisting of all matrices with determinant 1 is called the special unitary group and denoted by $SU(n)$. $U(n)$ and $SU(n)$ are subgroups of $GL(n, C)$ and $SL(n, C)$. They are both compact, connected Lie groups. $SU(1) = \{1\}$, and $U(1)$ is the classical multiplicative group of all complex numbers λ such that $|\lambda| = 1$. Classically, the center z of $U(n)$ consists of all diagonal matrices λI, $\lambda \in C$, $|\lambda| = 1$; $z \simeq U(1) \simeq S^1$ and $U(n)/SU(n) \simeq U(1)$, $z.\ SU(n) = U(n)$. For $n \geq 2$, $SU(n)$ is a simple, semisimple and simply connected Lie group. $PU(n) = U(n)/z$ is called the projective unitary group. $PU(n) \simeq SU(n)/z \cap SU(n)$, $z \cap SU(n) \simeq \mathbf{Z}/n\mathbf{Z}$ ($PU(n)$ is locally isomorphic to $SU(n)$). $U(n)$ and $SU(n)$ are compact Lie groups.

1.1.1.2 Table of Principal Subgroups of $GL(n, \mathrm{C})$—cf. Fig. 1.1

Interpretation: "$SL(n, \mathbf{C}) \rightarrow GL(n, \mathbf{C})$" means that $SL(n, \mathbf{C})$ is a subgroup of $GL(n, \mathbf{C})$. For a complex matrix $\alpha \in GL(n, \mathbf{C})$ we put

$$\alpha^{\sim} = ({}^t\alpha)^{-1} = {}^t(\alpha^{-1}).$$

For all $\alpha \in GL(n, \mathbf{C})$, $\alpha = \overline{\alpha}$ if and only if $\alpha \in GL(n, \mathbf{R})$, $\overline{\alpha} = \alpha^{\sim}$ iff $\alpha \in U(n, \mathbf{C})$, $\alpha = \overline{\alpha} = \alpha^{\sim}$ iff $\alpha \in O(n, \mathbf{R})$, $\alpha = \alpha^{\sim}$ iff $\alpha \in O(n, \mathbf{C})$.

1.1.1.3 Orthogonal groups

Thus, $O(n, \mathbf{C}) = \{\alpha \in GL(n, \mathbf{C}) : ({}^t\alpha)^{-1} = \alpha^{\sim} = \alpha\}$, $O(n, \mathbf{R}) = GL(n, \mathbf{R}) \cap U(n)$, $SO(n, \mathbf{R}) = SL(n, \mathbf{C}) \cap O(n, \mathbf{R})$. $SO(n, \mathbf{R})$, denoted also by $O^+(n, \mathbf{R})$, is a normal subgroup of $O(n, \mathbf{R})$ of index 2. $O(n, \mathbf{R})$ and $SO(n, \mathbf{R})$, often respectively denoted by $O(n)$ and $SO(n)$, are both compact Lie groups, and $SO(n)$ is the connected component of the identity in $O(n)$.

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to $Z_2 \oplus Z_2$ when $n \equiv 0 \pmod 4$), (Cf. E. Cartan, *Annali di Matematica*, t. 4, 1927, pp. 209–256). As found out by E. Cartan (E. Cartan, op. cit.), the simply connected representatives of the exceptional structures G_2, F_4, E_6, E_7, E_8 possess centers that respectively are cyclic of respective orders 1, 1, 3, 2, 1. Up to isomorphism, there exists only one group of respective structure G_2, F_4, E_8. At last we have the following classical isomorphisms (Cf. for example Armand Borel, *Collected Papers, Volume I*, Springer-Verlag, 1983, p. 363) Spin 3 $\simeq SpU(1)$, Spin 4 $\simeq SpU(1) \times SpU(1)$, Spin 5 $\simeq SpU(2)$, Spin 6 $\simeq SU(4)$. $F_4 \supset$ Spin 9 $\supset$ Spin 8 $\supset T$, where T is a four-dimensional torus, maximal in each of the other groups (A. Borel, op. cit., p. 380).

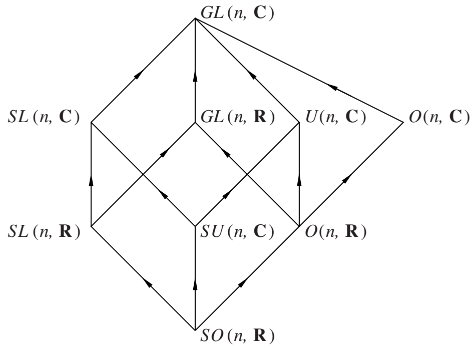

Fig. 1.1. Classical groups

$SO(n)$, $n \geq 3$, is a connected Lie group, and not a simply connected group. As mentioned before, $SO(n)$ possesses a twofold simply connected covering group denoted by Spin n. We recall that $SO(n, \mathbf{C})$, the subgroup of elements of $O(n, \mathbf{C})$ with determinant 1, is called the complex special orthogonal group.

1.1.2 Symplectic Groups: Classical Results

Let E be a $2n$-dimensional linear space over a field K, endowed with a bilinear non-degenerate skew-symmetric form [ ]: $(x, y) \in E^2 \to [x|y] \in K$. E is called a symplectic space over K. The group consisting of linear automorphisms of E that leave [|] invariant is called the symplectic group, denoted by $Sp(E)$.

Let E be K^{2n}; for all $X, Y \in E$, with respective coordinates x^j, y^k, with respect to the standard basis of K^{2n},

$$[X \mid Y] = \sum_{j=1}^{n} (x^j y^{j+n} - x^{j+n} y^j)$$

is called the standard symplectic product of K^{2n}. Then, $Sp(E)$ is called the standard symplectic group and denoted by $Sp(2n, K)$ in France, and often $Sp(n, K)$ in other countries. We choose the notation $Sp(2n, K)$. Any matrix in $Sp(2n, K)$ is always of determinant 1 and the center z of $Sp(2n, K)$ consists of I and $-I$. The quotient group of $Sp(2n, K)$ by its center z is called the projective symplectic group over K. For $n \geq 1$ and $K = \mathbf{R}, \mathbf{C}$, the group $PSp(n, K)$ is always simple.

1.1.3 Classical Algebraic Results

1.1.3.1 Classical Lie Algebras of Principal Subgroups of $GL(n, \mathbf{C})$

We recall the following classical results: Let E be a finite n-dimensional vector space over $\mathbf{R}$. Let $\mathcal{L}$(E) be the associated algebra of linear endomorphisms of V, and let

$GL(E) = \{x \in GL(E), \det x \neq 0\}$ be viewed as a Lie group. As usual, we denote the Lie algebra of $GL(E)$ by $gl(E)$.

We can identify $gl(E)$ with $\mathcal{L}(E)$ and we have $[X \mid Y] = XY - YX$, for all $X, Y \in gl(E)$. Therefore, the Lie algebra of $GL(n, \mathbf{R})$ is identical with $M(n, \mathbf{R})$, the classical algebra of all square real matrices of degree n. The dimension of $M(n, \mathbf{R})$ is n^2 over $\mathbf{R}$. In the same way we obtain the following list:[5]

- $GL(n, \mathbf{C})$ $gl(n, \mathbf{C}) \simeq M(n, \mathbf{C})$, dimension: $2n^2$ over $\mathbf{R}$.
- $SL(n, \mathbf{C})$ $sl(n, \mathbf{C}) \simeq \{X \in M(n, \mathbf{C}), \operatorname{Tr} X = 0\}$, dimension: $n^2 - 1$ over $\mathbf{C}$, $2(n^2 - 1)$ over $\mathbf{R}$.
- $U(n, \mathbf{C})$ $u(n, \mathbf{C}) \simeq \{X \in M(n, \mathbf{C}), {}^t\overline{X} = -X\}$ consisting of skew-hermitian matrices, dimension n^2 over $\mathbf{R}$.
- $O(n, \mathbf{C})$ $o(n, \mathbf{C}) \simeq \{X \in M(n, \mathbf{C}), {}^tX = -X\}$, consisting of complex skew-symmetric matrices, dimension: $n(n-1)$ over $\mathbf{R}$.
- $SU(n, \mathbf{C})$ $su(n, \mathbf{C}) \simeq \{X \in M(n, \mathbf{C}), {}^t\overline{X} = -X, Tr(X) = 0\}$ consisting of skew-hermitian matrices with null trace.
- $GL(n, \mathbf{R})$ $gl(n, \mathbf{R}) \simeq M(n, \mathbf{R})$, dimension n^2 over $\mathbf{R}$.
- $SL(n, \mathbf{R})$ $sl(n, \mathbf{R}) \simeq \{X \in M(n, \mathbf{R}), Tr(X) = 0\}$ consisting of real matrices with null trace, dimension $n^2 - 1$ over $\mathbf{R}$.
- $O(n, \mathbf{R})$ $o(n, \mathbf{R}) \simeq \{X \in M(n, \mathbf{R}), {}^tX = -X\}$, consisting of real skew-symmetric matrices with null trace, dimension: $n(n-1)/2$ over $\mathbf{R}$.
- $SO(n, \mathbf{R})$ $so(n, \mathbf{R}) \simeq \{X \in M(n, \mathbf{R}), {}^tX = -X\}$, consisting of real skew-symmetric matrices with null trace.

1.1.3.2 Other Groups and Their Lie Algebras

Let $U(p, q)$ be the group of matrices in $GL(p + q, \mathbf{C})$, which leave invariant the hermitian form:

$$z_1\overline{z}_1 + \cdots + z_p\overline{z}_p - z_{p+1}\overline{z}_{p+1} - \cdots - z_{p+q}\overline{z}_{p+q}$$
$$SU(p, q) = U(p, q) \cap SL(p + q, \mathbf{C}).$$

We remark that we have $U(n) = U(n, 0) = U(0, n)$ and $SU(n) = U(n) \cap SL(n, \mathbf{C})$, $SU^*(2n)$: the group of matrices in $SL(2n, \mathbf{C})$ which commute with the transformationψ of $\mathbf{C}^{2n}$ given by

$$(z_1, \ldots, z_n, z_{n+1}, \ldots, z_{2n}) \to (\overline{z}_{n+1}, \ldots, \overline{z}_{2n}, -\overline{z}_1, \ldots, -\overline{z}_n)$$

$SO(p, q)$: the group of matrices in $SL(p + q, \mathbf{R})$ which leave invariant thequadratic form

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2.$$

(We find again that $SO(n) = SO(0, n) = SO(n, 0)$.)

[5] This list found out by E. Cartan is given, pp. 339–359, in the following book: S. Helgason, *Differential Geometry and Symmetric Spaces*, 1962, Academic Press, New York and London.

$SO^*(2n)$ the group of matrices in $SO(2n, \mathbf{C})$ which leave invariant the skew-hermitian form

$$-z_1\overline{z}_{n+1} + z_{n+1}\overline{z}_1 - z_2\overline{z}_{n+2} + z_{n+2}\overline{z}_2 - \cdots - z_n\overline{z}_{2n} + z_{2n}\overline{z}_n$$

In the original Elie Cartan's list $Sp(2n, \mathbf{C})$ (denoted there by $Sp(n, \mathbf{C})$) is defined as the group of matrices in $GL(2n, \mathbf{C})$ which leave invariant the exterior form

$$z_1 \wedge z_{n+1} + z_2 \wedge z_{n+2} + \cdots + z_n \wedge z_{2n}$$

and $Sp(2n, \mathbf{R})$, denoted there by $Sp(n, \mathbf{R})$, is defined as the group of matrices in $GL(2n, \mathbf{R})$ which leave invariant the exterior form

$$x_1 \wedge x_{n+1} + x_2 \wedge x_{n+2} + \cdots + x_n \wedge x_{2n}$$

$SpU(p, q)$ the group of matrices in $Sp(2(p+q), \mathbf{C})$, or in $Sp(p+q, \mathbf{C})$ with Cartan's notations which leave invariant the hermitian form ${}^tZK_{pq}\overline{Z}$ where

$$K_{pq} = \begin{pmatrix} -I_q & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & -I_q & 0 \\ 0 & 0 & 0 & I_p \end{pmatrix}$$

By definition $SpU(n) = SpU(0, n) = SpU(n, 0)$ and $SpU(n) = Sp(2n, \mathbf{C}) \cap U(2n)$. The Lie algebras of these groups are respectively:

$u_{p,q} = \begin{pmatrix} Z_1 & Z_2 \\ {}^t\overline{Z}_2 & Z_3 \end{pmatrix}$ Z_1, Z_3 skew-hermitian of order q and p respectively, Z_2 arbitrary

$su_{p,q} = \begin{pmatrix} Z_1 & Z_2 \\ {}^t\overline{Z}_2 & Z_3 \end{pmatrix}$ Z_1, Z_3 skew-hermitian of order q and p respectively, $\mathrm{Tr}\, Z_1 + \mathrm{Tr}\, Z_3 = 0$, Z_2 arbitrary

$su^*(2n) = \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix}$ Z_1, Z_2 $n \times n$ complex matrices, $\mathrm{Tr}\, Z_1 + \mathrm{Tr}\, \overline{Z}_1 = 0$

$so(p, q) = \begin{pmatrix} X_1 & X_2 \\ {}^tX_2 & X_3 \end{pmatrix}$ All X_i real, X_1, X_3 skew-symmetric of order q and p respectively, X_2 arbitrary

$so^*(2n) = \begin{pmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{pmatrix}$ Z_1, Z_2 $n \times n$ complex matrices, Z_1 skew, Z_2 hermitian.

$sp(2n, \mathbf{C}) = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -{}^tZ_1 \end{pmatrix}$ Z_1, Z_2, Z_3 complex $n \times n$ matrices, Z_2 and Z_3 symmetric.

$sp(2n, \mathbf{R}) = \begin{pmatrix} X_1 & X_2 \\ X_3 & -{}^tX_1 \end{pmatrix}$ X_1, X_2, X_3 real $n \times n$ matrices, X_2 and X_3 symmetric.

$$spu(p,q) = \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ -{}^t\bar{Z}_{13} & Z_{22} & {}^tZ_{11} & Z_{21} \\ -\overline{Z}_{13} & \overline{Z}_{14} & \overline{Z}_{11} & -\overline{Z}_{12} \\ {}^t\overline{Z}_{14} & -\overline{Z}_{24} & -{}^tZ_{12} & \overline{Z}_{22} \end{pmatrix}$$

Z_{ij} complex matrix, Z_{11} and Z_{13} of order q, Z_{12} and Z_{14} $q \times p$ matrices, Z_{11} and Z_{22} are skew-hermitian, Z_{13} and Z_{24} are symmetric.

We recall the following result:

1.1.3.1.1 Theorem *The groups $SU(p,q)$, $SU^*(2n)$, $SO^*(2n)$, $Sp(2n,\mathbf{R})$, $SpU(p,q)$ are all connected. $SO(p,q)$, $0 < p < p+q$ has two connected components.*[6]

1.1.4 Classic Groups over Noncommutative Fields

1.1.4.1 Classic Results

Let E be a right linear space over a noncommutative field K. We recall that the set of all linear transformations of E becomes a group under the classical composition of mappings, called by definition the general linear group of E and denoted by $GL(E)$. Such a group is isomorphic to the multiplicative group of all invertible square matrices of degree n, with coefficients in K. The corresponding commutator subgroups, respectively denoted by $SL(E)$ and $SL(n,K)$, are called the special linear group of degree n on E and over K, respectively. The center z of $GL(n,K)$ is the set of all scalar matrices associated with nonzero elements in the center of K.

Let C be the commutator subgroup of the multiplicative group K^* of K. For $n \geq 2$, $GL(n,K)/SL(n,K)$ is isomorphic to K^*/C.[7] The center z_0 of $SL(n,K)$ is the set $\{\alpha I, \alpha^n \in C\}$.

The quotient group $PSL(n,K) = SL(n,k)/z_0$ is called the projective special linear group of degree n over K. Since K is a noncommutative field, if $n \geq 2$, $PSL(n,K)$ is always a simple group.

1.1.4.2 $U(n,K,f)$: Unitary Group Relative to an ε-Hermitian Form

We recall some basic results. Let K be a field (commutative or noncommutative). Let $\mathcal{J}$ be an antiautomorphism of K [for all $\alpha, \beta \in K$, $(\alpha+\beta)^{\mathcal{I}} = \alpha^{\mathcal{J}} + \beta^{\mathcal{J}}$, $(\alpha\beta)^{\mathcal{J}} = \beta^{\mathcal{J}}\alpha^{\mathcal{J}}$ and $\mathcal{J}$ is a bijection from K onto K]. $\mathcal{J}$ is called an involution of K. Let E be a right linear n-dimensional space on K. By definition, a sesquilinear form[8] relative to $\mathcal{J}$ is a mapping $f : E \times E \to K$ such that for all $x, x_1, x_2, y, y_1, y_2 \in E$, for all $\lambda, \mu \in K$,

[6] Helgason, op. cit. p. 346.

[7] One defines for $A \in GL(n,K)$ an element $\det A = K^*/C$ called the determinant of A, which leads to such an isomorphism. The theory developed by J. Dieudonné gives the ordinary case for a classical field K. (Cf. J. Dieudonné, *La Géométrie des Groupes Classiques*, op. cit.)

[8] We choose the definition given by J. Dieudonné, *La Géométrie des Groupes Classiques*, op. cit., p. 10.

1. $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$;
2. $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$;
3. $f(x\lambda, y) = \lambda^{\mathcal{J}} f(x, y)$;
4. $f(x, y\mu) = f(x, y)\mu$.

If, moreover, $f(y, x) = f(x, y)^{\mathcal{J}}$, then f is called a hermitian form relative to $\mathcal{J}$. If $f(y, x) = -f(x, y)^{\mathcal{J}}$, f is called a skew-hermitian form relative to $\mathcal{J}$. If $\mathcal{J} = 1_K$, then a hermitian form is a symmetric bilinear form, and a skew-hermitian form is an antisymmetric bilinear form.

A linear space E endowed with a nondegenerate hermitian form f is called a hermitian linear space, and $f(x, y)$ is called the hermitian inner product of $x, y \in E$.

Suppose now $\mathcal{E} = \pm 1$ and K is a field of characteristic zero and let $\mathcal{A} = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$ (or more generally a division algebra D over K) with center$\mathcal{A}_1$ and with $[\mathcal{A}_1 : K] = d$, $[D : \mathcal{A}_1] = r^2$.[9] Let E be an n-dimensional linear space over K with the structure of a right $\mathcal{A}$-module. Let us assume that $\mathcal{A}$ has a K-linear involution $\mathcal{J}$. An $\mathcal{A}$-valued $\mathcal{E}$-hermitian form f with respect to $\mathcal{J}$ is by definition a map f from $E \times E \to$ K such that $f(x, y\lambda) = f(x, y)\lambda$ and $f(y, x) = \mathcal{E} f(x, y)^{\mathcal{J}}$, $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$, $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$.

Let $\mathcal{A}$ denote the algebra of D-linear transformations of V. For a fixed basis of E we have $\mathcal{A} \simeq M(n, D)$. Let $e = \{e_1, \ldots, e_n\}$ be a fixed basis of E. Then f may be expressed by F, a hermitian square matrix of degree n such that $f(x, y) = ({}^t X)^{\mathcal{J}} F Y$, where X, Y, F are respective the matrices of x, y, f relative to the basis e. To any nondegenerate $\mathcal{A}$-$\mathcal{E}$-hermitian form f we can associate an involution$*$, namely its relative adjunction classically defined as follows: for any linear operator a of $\mathcal{A}$, $f(ax, y) = f(x, a^* y)$, or in matrix notation, if A is the matrix of a relative to e, then $A^* = (H^{-1})^t A^{\mathcal{J}} H$. This result will be used later.

An involution $\mathcal{J}$ is of the first kind if it fixes all elements in the center of the algebra, and of the second kind otherwise.

We have

$$\dim_{\mathcal{A}_1^+} \mathcal{A}^+ = \begin{cases} \frac{1}{2}(r^2 + \eta r) & \text{if } \mathcal{J} \text{ is of the first kind,} \\ r^2 & \text{if } \mathcal{J} \text{ is of the second kind,} \end{cases}$$

where $\mathcal{A}^{\pm} = \{a \in \mathcal{A} | a^{\mathcal{J}} = \pm a\}$ and $\mathcal{A}_1^{\pm} = \mathcal{A}^{\pm} \cap \mathcal{A}_1$ and $\eta = \pm 1$ is the sign of the involution $\mathcal{J}$. The sign of an involution $*$ of $\mathcal{A}$ is defined similarly. We will often write that the sign of $*$ is $= 0$ if $*$ is of the second kind.

One can verify that if $\mathcal{J}$ is of the first kind with sign η and if $*$ is defined by an $(\mathcal{A}, \mathcal{E})$-hermitian form relative to $\mathcal{J}$, then $*$ is of the first kind with sign $\mathcal{E}\eta$.

Let $\mathcal{A}$ denote the algebra of D-linear transformations on E. For the fixed basis $e = \{e_i\}$, one has $\mathcal{A} \simeq M(n, D)$. We define $\mathcal{A}^{\times} = GL(E \mid \mathcal{A}) =$ {the multiplicative group of units of $\mathcal{A}$},

$$\mathcal{A}^{(1)} = \{a \in \mathcal{A}^{\times} / N(a) = 1\} = SL(E \mid D),$$

[9] When K' is an extension field of a field K, the "degree" of the extension is denoted by $[K':K]$, and when $K' \mid K$ is a Galois extension, the Galois group is denoted by $\mathrm{Gal}(K' \mid K)$.

where a unit is an invertible element and N denotes the reduced norm relative to its centerK_1. The corresponding matrix groups are respectively denoted by $GL(n, \mathcal{A})$ and $SL(n, \mathcal{A})$. The Lie algebra of $SL(n, \mathcal{A})$ is $sl(n, \mathcal{A}) = \{X \in M(n, \mathcal{A})/\mathrm{Tr}\, X = 0\}$, where Tr is the reduced trace of $\mathcal{A}$ relative to the center K_1 of K.[10]

We define the unitary group and the special unitary group by

$$\begin{aligned} U(E,h) &= \{a \in GL(E|\mathcal{A})/h(ax, ay) = h(x,y), (x,y) \in E^2\} \\ &= \{a \in GL(E|\mathcal{A})/A^{\mathcal{J}} A = 1\}, \\ SU(E,h) &= U(E,h) \cap SL(E|\mathcal{A})\}. \end{aligned}$$

The corresponding matrix groups are classically respectively denoted by $U(n, \mathcal{A}, h)$, and $SU(n, \mathcal{A}, h)$. For instance,

$$SU(n, \mathcal{A}, h) = \{\mathcal{A} \in SL(n, \mathcal{A})/{}^t A^{\mathcal{J}} H A = H\}.$$

The corresponding Lie algebra is

$$su(n, \mathcal{A}, h) = \{X \in sl(n, \mathcal{A})/{}^t X^{\mathcal{J}} + X = 0\}.$$

When $\mathcal{A} = K$, an $\mathcal{E}$-hermitian form is called $\mathcal{E}$-symmetric, i.e., symmetric or alternating according as the sign $\mathcal{E} = 1$ or -1, and the corresponding unitary group is called an orthogonal group or symplectic group. In this case the letter is respectively replaced by O or Sp.

More precisely, we recall that for $\mathcal{J} = 1$ and $\mathcal{E} = 1$ a unitary transformation is called an orthogonal transformation and the corresponding group is then denoted by $O(n, K, f)$. For $\mathcal{J} = 1$ and $\mathcal{E} = -1$, in the same way, we obtain a symplectic transformation, and the corresponding symplectic group is denoted by $Sp(2n, K)$ (or often $Sp(n, K)$). (One can verify that in these cases, the groups associated with different choices of f are mutually isomorphic.) Let f be an $\mathcal{E}$-hermitian form on E; f is called an $\mathcal{E}$-trace form $|f|$ if for all $x \in E$, there exists $\lambda \in K$ such that $f(x,x) = \lambda + \mathcal{E}\lambda^{\mathcal{J}}$. If $\mathcal{J} = 1$ and $\mathcal{E} = -1$ (K commutative) or $\mathcal{E} = 1$ and K is not of characteristic 2, then any $\mathcal{E}$-hermitian form is an $\mathcal{E}$-trace form.

The classic Witt theorem can be proved in the following way: If f is an $\mathcal{E}$-trace form, a linear mapping v of any subspace F of E into E such that for all $x, y \in F$, $f(v(x), v(y)) = f(x,y)$, can be extended to an element u of the unitary group $U(n, K, f)$ associated with f. Thus, $U(n, K, f)$ acts transitively on the maximal totally isotropic subspaces, and their common dimension is the index m of f. If the field K is the classic skew field $\mathbf{H}$ (or more generally a quaternion algebra over a Pythagorean ordered field P) and f is a skew-hermitian form, according to a result of J. Dieudonné, there exists an orthogonal basis e_i of E such that $f(e_i, e_i) = j$

[10] The following definitions can be found in A. A. Albert, Structure of algebras, op. cit., p. 122. Let $\mathcal{A}$ be an algebra over K, $\mathcal{L}$ an algebraically closed scalar extension of K. Let $\wedge$ be any ν-rowed representation $a \to a^*$ of $\mathcal{A}$ by $\mathcal{A}^*$. The determinant of a^* is called the reduced norm $N_\wedge(a)$ and the sum of the diagonal elements of a^* is called the reduced trace $T_\wedge(a)$, for any $a \in \mathcal{A}$.

(quaternion unit), $1 \leq i \leq n$. We will use this result later.[11] (In this case, the unitary corresponding group $U(n, K, f)$ is determined by only n and K.)

1.1.4.3 Results Concerning the Cases of $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$

Then $GL(n, K)$, $SL(n, K)$, and $U(n, K, f)$ are all Lie groups, and $SL(n, K)$ and $U(n, K, f)$ are simple Lie groups except for the following cases:

(α) $n = 1$, $K = \mathbf{R}$ or $\mathbf{C}$,

(β) $n = 2$, $K = \mathbf{R}$, $\mathcal{J} = 1$, $\mathcal{E} = 1$

(γ) $n = 4$, $K = \mathbf{R}$ or $\mathbf{C}$, $\mathcal{J} = 1$, $\mathcal{E} = 1$, $m = 2$.

In cases (α), (β), they are commutative groups; in case (γ), they are locally direct sums of two noncommutative simple groups.

1.1.4.4 Case of $K = \mathbf{H}$

H contains **C** as a subfield and a vector space E of dimension n over **H** has the structure of a vector space of dimension $2n$ over **C**. Thus, $GL(n, \mathbf{H})$ can be considered as a subgroup of $GL(2n, \mathbf{C})$ in a natural way.

Real Forms of $GSL(n, \mathbf{C})$, $SO(n, \mathbf{C})$, $Sp(2n, \mathbf{C})$

Each of these classical simple groups has the structure of an algebraic classical simple group defined over **R**. The real forms of G, i.e., the algebraic subgroups of G, the scalar extension of which to **C** is G, can be viewed as $SL(n, K)$, $U(n, K, f)$ corresponding to $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$.

A real form of a complex classical group G is conjugate in G to one of the following groups:

(i) The real forms of $SL(n, \mathbf{C})$: $SL(n, \mathbf{R})$ (type AI), $SL(k, \mathbf{H})$, only for $n = 2k$, (type AII), and the special unitary group $SU(n, m, \mathbf{C})$, $0 \leq m \leq [n/2]$, relative to a hermitian form of index m (type AIII). ($[x]$ denotes the integer part of the real number x.)

(ii) The real forms of $SO(2n+1, \mathbf{C})$: the proper orthogonal group $SO(2n+1, m, \mathbf{R})$, $0 \leq m \leq n$ (type BI), relative to a quadratic form of index m on a space of dimension $2n + 1$.

(iii) The real forms of $SO(2n, \mathbf{C})$: $SO(2n, m, \mathbf{R})$, $0 \leq m \leq n$ (type DI), and $U(n, \mathbf{H}, f)$ relative to a skew-hermitian form f on **H** (type DIII).

(iv) The real forms of $Sp(2n, \mathbf{C})$: $Sp(2n, \mathbf{R})$, type CI, the unitary group $U(2n, m, \mathbf{H})$, $0 \leq m \leq n$, relative to a hermitian form of index m on **H** (type CII); and $SpU(n)$—often denoted by $Sp(n)$—corresponds to the special case $m = 0$.

[11] Dieudonné J., *La Géométrie des Groupes Classiques*, op. cit., p. 16.

The quotient groups of these real forms by their centers can be realized[12] as the groups of automorphisms of semisimple algebras with involutions $\mathcal{J}$ that commute with $\mathcal{J}$.

1.2 Clifford Algebras

As pointed out by N. Bourbaki,[13] in 1876 William Clifford introduced the algebras known as Clifford algebras and proved that they are tensor products of quaternion algebras or of quaternion algebras by a quadratic extension.[14] Let us first recall some classical results concerning quaternion algebras.

1.2.1 Elementary Properties of Quaternion Algebras

1.2.1.1 Definition Let K be a field of characteristic different from 2. A quaternion algebra $\mathcal{A}$ over K is, by definition, a central simple associative algebra over K with $[\mathcal{A} : K] = 4$. If $\mathcal{A}$ is not a division, one has $\mathcal{A} \simeq M(2, K)$, in which case $\mathcal{A}$ is called a "split" quaternion algebra.

Let $a_1, a_2 \in K^\times$; one can define a quaternion algebra $A(a_1, a_2)$ as an algebra with unit element 1 over K generated by two elements e_1, e_2 that satisfy the following relations: $e_1^2 = a_1$, $e_2^2 = a_2$, $e_1e_2 = -e_2e_1$. As usual, we set $e_0 = 1$, $e_3 = e_1e_2$, $a_3 = -a_1a_2$. Then $\{e_0 = 1, e_1, e_2, e_3\}$ is a basis of $A(a_1, a_2)$ over K with the following table of multiplication:

first factor	second factor		
	e_1	e_2	e_3
e_1	a_1	e_3	a_1e_2
e_2	$-e_3$	a_2	$-a_2e_1$
e_3	$-a_1e_2$	a_2e_1	a_3

where $e_i^2 = a_ie_0$, $(1 \le i \le 3)$ and $e_ie_j = -e_je_i$. $A(a_1, a_2)$ is often denoted by $(\frac{a_1, a_2}{K})$. We have the following statement.[15]

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[12] A. Weil, Algebras with involutions and the classical groups, *Collected Papers, Vol. II*, pp. 413–447 Springer-Verlag, New York, 1980.

[13] N. Bourbaki, *Eléments d'Histoire des Mathématiques*, Hermann, Paris 1969, p. 173.

[14] W. K. Clifford, *Mathematical Papers*, London, Macmillan, 1882, pp. 266–276. This fact can be classically illustrated by the construction due to Brauer and Weyl of the Clifford algebra associated with a standard complex regular space, which is for n even, $n = 2r$, isomorphic to $m(2^r, \mathbf{C})$, the total matrix algebra of degree 2^r with coefficient in $\mathbf{C}$, and for n odd, $n = 2r + 1$ isomorphic to the direct sum $m(2^r, \mathbf{C}) \bigoplus m(2^r, \mathbf{C})$ of two copies of such an algebra (cf. exercises below).

[15] Cf. I. Satake, op. cit. pp. 270–273.

1.2.1.2 Proposition *Let $\mathcal{A}$ be a quaternion algebra over K. There exists a unique involution $\mathcal{J}_0$ of $\mathcal{A}(a_1, a_2), q \mapsto q^{\mathcal{J}_0}$ of the first kind satisfying the following mutually equivalent conditions:*

(1) $\{q \in \mathcal{A} \mid q^{\mathcal{J}_0} = q\} = K$.
(2) *The sign of $\mathcal{J}_0$ is -1.*
(3) *The reduced trace of $q \in \mathcal{A}$ is given by $\mathrm{Tr}(q) = q^{\mathcal{J}_0} + q$.*
(4) *The reduced norm $N(q)$ of $q \in \mathcal{A}$ is $N(q) = qq^{\mathcal{J}_0}$. In the case of $\mathcal{A}(a_1, a_2)$ for $q = e_0\alpha_0 + e_1\alpha_1 + e_2\alpha_2 + e_3\alpha_3$, $q^{\mathcal{J}_0} = e_0\alpha_0 - e_1\alpha_1 - e_2\alpha_2 - e_3\alpha_3$ and*

$$N(q) = qq^{\mathcal{J}_0} = \alpha_0^2 - \sum_{i=1}^{3} a_i\alpha_i^2.$$

It is well known that $\mathcal{C}l(\mathcal{A})$ denotes the Brauer class of $\mathcal{A}$[16] and if ${}_2B(K)$ denotes the subgroup of the Brauer group $B(K)$ consisting of all elements of order at most two, $\mathcal{C}l(\mathcal{A}(a_1a_2))$ leads to a bilinear pairing

$$K^\times/(K^\times)^2 \times K^\times/(K^\times)^2 \rightarrow_2 B(K).$$

Moreover, $\mathcal{C}l(\mathcal{A}) = 1$, ($\mathcal{A}(a_1, a_2)$ is a "split" quaternion algebra), iff the equation $a_1x_1^2 + a_2x_2^2 = 1$ has a solution in K. Thus, the classical real quaternion algebra

[16] We recall some *classical definitions* (T. Y. Lam, The algebraic theory of quadratic forms, op. cit. chapter 4 for example): Let A be a finite-dimensional algebra over a field K: briefly we will call it a K-algebra. Let S be a subset of a K-algebra; $C_A(S) = \{a \in A : as = sa$, for all $s \in S\}$ is called by definition the centralizer of S: $C_A(A) = Z(A)$ is the center of A. A is called K-central (or central over K) iff its center $\mathbf{Z} = K1$. A is called simple iff A has no proper two-sided ideals. A is called a central simple algebra (C.S.A.) over K iff A is both K-central and simple. We have the following statements:

Theorem. *If A, B are K-algebras, and $A' \subset A$, $B' \subset B$ are subalgebras, then $\mathcal{C}_{A\otimes B}(A' \otimes B') = \mathcal{C}_A(A') \otimes \mathcal{C}_B(B')$. If A, B are K-central, $A \otimes B$ is K-central. If A is a C.S.A. over K and B a simple algebra, $A \otimes B$ is simple. If A, B are both C.S.A. over K, $A \otimes B$ is C.S.A. over K.*

Definition. *Let A, A' be both C.S.A. over K. A is similar to A' if there exist finite-dimensional spaces V and V' such that $A \otimes \mathrm{End}V \simeq A' \otimes \mathrm{End}V'$ as K-algebras. This relation of similarity is an equivalence relation. The set of similarity classes of C.S.A.s becomes a semigroup with $[K] = [M(n, K)]$ as the identity, denoted by $B(F)$.*

Proposition and Definition. *For any K-algebra A, let A^0 denote the opposite algebra. If A is a C.S.A., A^0 is a C.S.A. and $A \otimes A^0 \simeq \mathrm{End}\,A$ (algebra of linear endomorphisms of A). In particular, $B(F)$ is an abelian group with $[A]^{-1} = [A^0]$ for any C.S.A. A. $B(F)$ is called the Brauer group of A. C.*

C. T. C. Wall, as clearly pointed out by T. Y. Lam (op. cit. pp. 95–96), first "observed that it is possible and (expedient) to define a "graded Brauer group" using similarity classes of central simple graded K-algebras (CSGA). Wall's 'graded Brauer group' has since been known as the Brauer–Wall group (written BW(F))." Example (T. Y. Lam, op. cit, p. 117): $BW(R) = \mathbf{Z}/8\mathbf{Z}$.

$\mathbf{H} = \mathcal{A}(-1, -1)$, often denoted by $(\frac{-1,-1}{\mathbf{R}})$, is the unique division quaternion algebra over the real field $\mathbf{R}$.

When K is a local field or an algebraic number field of finite degree, the previous pairing is surjective and complete. Therefore, a central division algebra B over such a field K with an involution $\mathcal{J}$ of the first kind is necessarily a quaternion algebra, and any involution of B of the first kind with sign η can be written as $q \to f^{-1} q^{\mathcal{J}} f$, where f belongs to the multiplicative group of units of B and $f^{\mathcal{J}} = -\eta f$.

Let $B = (a, b/K)$ be a quaternion division algebra over K and let $K' = K(\sqrt{a})$. Then we have an isomorphism $B \otimes_K K' \simeq M(2, K')$ determined by

$$M(e_1) = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix},$$
$$M(e_2) = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}.$$

Let us denote by e_{ij} the corresponding units in $B \otimes_K K'$:

$$e_{11} = \frac{1}{2}\left(1 + \frac{1}{\sqrt{a}} e_1\right), \qquad e_{12} = \frac{1}{2b}\left(e_2 + \frac{1}{\sqrt{a}} e_1 e_2\right),$$
$$e_{21} = \frac{1}{2}\left(e_2 - \frac{1}{\sqrt{a}} e_1 e_2\right), \qquad e_{22} = \frac{1}{2}\left(1 - \frac{1}{\sqrt{a}} e_1\right).$$

Let $\mathrm{Gal}(K' \mid K) = \{1, l_0\}$, where l_0 is the nontrivial automorphism of K' over K determined by $\sqrt{a}^{l_0} = -\sqrt{a}$. We have the following relations: $e_{12}^{l_0} = e_{22}$, $e_{12}^{l_0} = b^{-1} e_{21}$, and

$$M(D) = \left\{ \begin{pmatrix} u & bv \\ -v^{l_0} & u^{l_0} \end{pmatrix} \Big/ u, v \in K' \right\}.$$ [17]

1.2.2 Clifford Algebras

1.2.2.1 Definitions and Basic Results

Let K be a field of characteristic different from 2. Let E be a vector space of dimension n over K. Let q denote a regular quadratic form on E and let B be the corresponding nondegenerate symmetric bilinear form such that for any $x \in E$, $q(x) = B(x, x)$. (E, q) is called a regular quadratic space. Hence, we have $q(x) = B(x, x) = {}^t X B X$, where X and B denote respectively the matrices of $x \in E$ and of B with respect to a

[17] Thus we find the classical representation of $\mathbf{H} = (\frac{-1,-1}{\mathbf{R}})$ as the real algebra of matrices

$$\mathbf{H} = \begin{pmatrix} u & -v \\ \overline{v} & \overline{u} \end{pmatrix},$$

where for $v = \beta + i\gamma$, $v^{l_0} = \overline{v} = \beta - i\gamma$ is the classical conjugate of v (cf. exercises).

given basis e of E. As is well known, there exists an orthogonal basis $e = \{e_1, \ldots, e_n\}$ *of* E *such that for any*

$$x = \sum_{i=1}^{n} e_i x_i \in E,$$

we have

$$q(x) = \sum_{i=1}^{n} a_i x_i^2,$$

or equivalently, $B(e_i, e_j) = \delta_{ij} a_i$ $(1 \leq i, j \leq n)$*. By definition,*

$$\triangle(q) = (-1)^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} a_i \ (\mathrm{mod}(K^{\times})^2)$$

is the "discriminant" of q.

The construction of a Clifford algebra associated with a quadratic regular space (E, q) is based on the fundamental idea of taking the square root of a quadratic form, more precisely of writing $q(x)$ as the square of a linear form φ on E such that for any $x \in E, q(x) = (\varphi(x))^2$.

1.2.2.2 Clifford Mappings

Let A be any associative algebra with a unit element 1_A.

1.2.2.2.1 Definition A Clifford mapping f from (E, q) into A is a linear mapping f such that for any $x \in E$, $(f(x))^2 = q(x)1_A$. By polarization, we obtain $f(x)f(y) + f(y)f(x) = 2B(x, y)1_A$, for any $x, y \in E$.

1.2.2.3 Clifford Algebra $C(E, q)$

1.2.2.3.1 Definition For a given quadratic regular space (E, q) we define a Clifford algebra associated with (E, q) to be any pair (C, f_C), where C is an associative algebra over K with a unity 1_C and f_C is a Clifford mapping from (E, q) into C such that:

(1) 1_C and $f_C(E)$ linearly generate C.
(2) For any Clifford mapping f from (E, q) into the associative algebra A with unity 1_A, there exists an algebra homomorphism F from C into A such that $f = F \circ f_C$.

We recall the following classical theorems:

1.2.2.3.2 Theorem *Any quadratic regular space* (E, q) *possesses a Clifford algebra which can be defined as the quotient of the tensor algebra* $\mathcal{T}(E)$ *of* E_1 *by the two-sided* *ideal $I(q)$ of $\mathcal{T}(E)$ generated by the elements $x \otimes x - q(x)1$, for any $x \in E$. The resulting quotient associative algebra $\mathcal{T}(E)/I(q)$ is then denoted by $C(E, q)$ and called the Clifford algebra of the quadratic regular quadratic space (E, q). The composite of the canonical injective mapping $E \to \mathcal{T}(E)$ and of the projection $\mathcal{T}(E) \to C(E, q)$ is a linear injection $f_C : E \to C(E, q)$. It becomes a Clifford mapping from E into $C(E, q)$ and leads to the identification of E with $f_C(E)$.*

If the dimension of E over K is n, then $C(E, q)$ is 2^n-dimensional over K. If $\{e_1, \ldots, e_n\}$ is a basis of E, then $1, e_i, e_i e_j, (i < j), \ldots, e_1 e_2 \cdots e_n$, form a basis of $C(E, q)$. In particular, if $\{e_i\}_{1 \le i \le n}$ is an orthogonal basis of E relative to q, we have

(α) $e_i e_j = -e_j e_i$, $(e_i)^2 = q(e_i)1$ ($i, j = 1, 2, \ldots, n, i \neq j$). (Furthermore, $x^2 = q(x)1_C$ for any $x \in E$.) In this case $C(E, q)$ may be defined as an associative algebra (with a unit element) generated by the $\{e_i\}$ together with the relations (α).[18]

The case $q = 0$ leads to $C(E, q = 0) \simeq \wedge E$ (the Grassmann or exterior algebra over E).

1.2.2.4 The Principal Automorphism π and the Principal Antiautomorphism τ of $C(E, q)$

1.2.2.4.1 Theorem *There exists a unique automorphism π of the algebra $C(E, q)$ such that $\pi(x) = -x$ for any $x \in E$. This automorphism π is called the principal automorphism of $C(E, q)$, and $\pi^2 = 1$.*

There exists a unique antiautomorphism τ of the algebra $C(E, q)$ such that $\tau(x) = x$ for any $x \in E$. This antiautomorphism τ is called the principal antiautomorphism of $C(E, q)$, and we have $\tau^2 = 1$.

$\nu = \pi \circ \tau = \tau \circ \pi$ is the unique antiautomorphism of $C(E, q)$ such that for any $x \in E$, $\nu(x) = -x$. For any $a \in C(E, q)$ we often write $\nu(a) = a^*$ or $\widetilde{a}$; ν is often called the conjugation of $C(E, q)$.

1.2.2.5 The Even Clifford Algebra $C^+(E, q)$

1.2.2.5.1 Theorem *Let $e = \{e_1, \ldots, e_n\}$ be an orthogonal basis relative to q. With the above notation, we have the following relations:*

$$\begin{cases} (e_i)^2 = a_i & (1 \le i \le n), \\ e_i e_j + e_j e_i = 0 & (1 \le i, j \le n, i \neq j). \end{cases}$$

We put $C^+ = \{e_{i_1} \cdots e_{i_m} (i_1 < \cdots < i_m), m \text{ even}\}_K$ (i.e., the linear space over K generated by the $(e_{i_1} \cdots e_{i_m})$, m even), $C^- = \{e_{i_1} \cdots e_{i_m} (i_1 < \cdots < i_m), m \text{ odd}\}_K$.

Since the two-sided ideal $I(q)$ is generated by "even" elements, the definition of $C^\pm$ is independent of the basis.

[18] Regularity is not required in the definitions above, but only quadratic regular spaces will be considered.

$C = C^+ \oplus C^-$ as a vector space, $(C^+)^2 = (C^-)^2 = C^+, C^+C^- = C^-C^+ = C^-$. Thus $C(q) = C^+ \oplus C^-$ has the structure of a graded algebra with the index group $\{\pm 1\}$. The elements of C^+ (respectively C^-) are called respectively even elements and odd elements. C^+ and C^- are both linear subspaces of $C(E, q)$ with the same dimension 2^{n-1}, as respective eigenspaces of $C(E, q)$ for the eigenvalue 1, respectively -1, of the principal automorphism π of $C(E, q)$. C^+ is a subalgebra of C with the same unit element $1_{C(E,q)}$.

If $q \neq 0$, the subalgebra C^+ of C can be expressed as the Clifford algebra of any subspace $E_1 = u^\perp$ of E, the orthogonal space of a regular vector u for the quadratic form $q_1 = -q(u)q$. For such a structure of C^+ the conjugation ν_1 is the restriction of the conjugation ν of C, and the principal automorphism π_1 is the inner automorphism of C^+ directed by u.

If for a fixed $p \in \mathbf{N}$ we call C_p the $\binom{n}{p}$-dimensional subspace of C spanned by the products $e_A = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_p}$, $1 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_p \leq n$, with exactly p factors, then C is the direct sum of the subspaces C_p. C_0 is identified with the field K, and C_1 with the vector space E. Thus[19,20]

$$C^+ = \sum_{p \text{ even}} C_p, \qquad C^- = \sum_{p \text{ odd}} C_p.$$

1.2.2.5.2 Proposition *C and C^+ are semisimple algebras over K, and the centers of C and C^+ are given as follows:*

	Cent C	Cent C^+
n even	K	$\{1, e_N\}_K$
n odd	$\{1, e_N\}_K$	K

where $e_N = e_1 \cdots e_n$.[21]

Then C (n even) and C^+ (n odd) are central simple algebras over K.

The anticenter A of $C(E, q)$ is defined as the linear space of the elements a of $C(E, q)$ that anticommute with any $x \in E$, or equivalently, that commute with even elements of C and anticommute with odd elements of $C(E, q)$.

If n is odd, $A = \{0\}$. If n is even $A = Ke_N$ with $e_N = e_1 \cdots e_n$. Furthermore, C (n even) and C^+ (n odd) both are in the Brauer class of $\otimes_{i<j}((-1)^{i+1}a_i, (-1)^j a_j)$.

Moreover, $[1, e_N]_K$ is a field if and only if $e_N^2 = \Delta(q) \notin (K^\times)^2$, and if n is odd, C is then a central simple algebra over the field extension $\tilde{K} = K[\sqrt{\Delta(q)}]$ of the field K.

Thus, if $\Delta(q) \notin (K^\times)^2$, then C (n odd) and C^+ (n even) are simple. Otherwise, they are the direct sum of two isomorphic central simple algebras. In either case, it

[19] Cf. for example, R. Deheuvels, *Tenseurs et Spineurs*, op. cit., pp. 235–238; or I. Satake, *Algebraic Structures of Symmetric Domains*, op. cit., pp. 231–287.

[20] The grading $C = \sum_p C_p = C^+ \oplus C_{-1}$ moreover $C = C^+ \oplus C_-$ is a graded Z_2 algebra.

[21] For a subset A of K-vector space E, $A_K = \{\cdots\}$ denotes the linear subspace of E generated by A.

is known that $C \sim C^+$ over $K(\sqrt{\Delta(q)})$ where the meaning of $\sim$ is that all simple components of both sides belong to the same Brauer class over $K(\sqrt{\Delta(q)})$.

More precisely, we have the following statement:

1.2.2.5.3 Theorem

- *If $\dim_K E$ is even, the Clifford algebra $C(E,q)$ is a central simple algebra over K. The principal automorphism π is then an inner automorphism of C.*
- *If $\dim_K E$ is odd and if $e_N^2 = \Delta(q) \notin (K^\times)^2$, then $C(E,q)$ is a central simple algebra over $\tilde{K} = K(\sqrt{\Delta(q)}) = \{\lambda 1_C + \mu e_N / \lambda, \mu \in K\}$, a quadratic extension of K, and $C = C^+ \oplus e_N C^+$ is an extension of C^+. Finally, $\pi(\lambda 1_C + \mu e_N) = \lambda 1_C - \mu e_N$ is the unique automorphism of $\tilde{K}$ different from the neutral element, which leaves K invariant in the Galois group of $\tilde{K}$:*

$$\pi(a_+ + e_N b_+) = a_+ - e_N b_+ \text{ with } a_+, b_+ \in C^+.$$

- *If $\dim_K E$ is odd and if $e_N^2 = \Delta(q) \in (K^\times)^2$, then $C(E,q)$ is the direct sum of two isomorphic central simple algebras both isomorphic to C^+. Let $e_N^2 = \alpha^2 \in K$. Let $\mathcal{E}_1 = \frac{1}{2}(1 + \frac{1}{\alpha} e_N)$ and $\mathcal{E}_2 = \frac{1}{2}(1 - \frac{1}{\alpha} e_N)$. We have $\mathcal{E}_1^2 = \mathcal{E}_1$, $\mathcal{E}_2^2 = \mathcal{E}_2$, $\mathcal{E}_1\mathcal{E}_2 = \mathcal{E}_2\mathcal{E}_1 = 0$, $\mathcal{E}_1 + \mathcal{E}_2 = 1$ and $e_N = \alpha(\mathcal{E}_1 - \mathcal{E}_2)$.*

$C = C^+\mathcal{E}_1 \oplus C^+\mathcal{E}_2$, the two components are both simple algebras, isomorphic to C^+, and π is the automorphism of C that interchanges the units $\mathcal{E}_1$ and $\mathcal{E}_2$ and leaves invariant the elements of C^+, and then interchanges $C_1 = C^+\mathcal{E}_1$ and $C_2 = C^+\mathcal{E}_2$.

Classical example:

Let us assume that $K = \mathbf{R}$ and that the signature of q is (r,s).

1.2.2.5.4 Proposition

$$C^+(r,s) \sim \begin{cases} \mathbf{R} \\ \mathbf{C} \\ \mathbf{H} \end{cases}$$

according as

$$r - s \equiv \begin{cases} 0, \pm 1 \\ \pm 2 \\ \pm 3, 4 \end{cases} \pmod 8$$

Since $e_N^\tau = (-1)^{\frac{n(n-1)}{2}} e_N$, the principal antiautomorphism τ is of the first kind for $C(r,s)$ if and only if $n \equiv 3 \pmod 4$ and for $C^+(r,s)$ if and only if $n \equiv 2 \pmod 4$. As pointed out by I. Satake, when τ is of the first kind, the sign $\varepsilon\eta$ can be determined by an easy computation of the dimension of the subspace of τ-stable elements.

We have the following statement:[22]

1.2.2.5.5 Proposition *When $C^+(r,s)$ is not simple, i.e., when n is even and $\Delta(q) \in (K^\times)^2$, either (in the case $n \equiv 2 \pmod 4$) τ interchanges the two simple components*

[22] I. Satake: *Algebraic Structures of Symmetric Domains*, op. cit, pp. 280–281.

of $C^+(r,s)$ *or (in the case* $n \equiv 0$ *(mod 4))* τ *leaves the simple components fixed and induces on each of them an involution of the first kind with the same sign.*

1.2.2.6 The Clifford Groups

1.2.2.6.1 Definitions

Let G be the set of all invertible elements g in $C(E,q)$ such that $gEg^{-1} = E$. Then G forms a group relative to the multiplication of $C(E,q)$. This group G is called the Clifford group. The subgroup $G^+ = G \cap C^+(E,q)$ is called the special Clifford group.

The linear transformation $\varphi(g) : x \to gxg^{-1}$ of E induced by $g \in G$ belongs to the orthogonal group $O(q)$ of E relative to q. Furthermore, the mapping $g \to \varphi(g)$ is a homomorphism from G into $O(q)$. Therefore, φ is a representation of G on E. This representation is called the *vector representation* of G. The kernel of φ is the set of invertible elements in the center Z of $C(E,q)$. If $x \in E \cap G$, then $q(x) \neq 0$ and $-\varphi(x)$ is the classical reflection mapping of E relative to $x^\perp$, the hyperplane orthogonal to x.

If $n = \dim E$ is even, $G = G^+ \cup G^-$, $\varphi(G^+) = SO(q) = O^+(q)$, $\varphi(G^-) = 0^-(q)$. G^+ is a subgroup of index 2 in G.

If $n = \dim E$ is odd, $\varphi(G) = \varphi(G^+) = SO(q)$. Any elements $g \in G$ can be written as $g = za_1 \cdots a_{2p}$ with $a_i, 1 \leq i \leq 2p$, regular vectors of E and $z = \alpha 1 + \beta e_N$. We note that $g \in G^+$ iff $\beta = 0$ and $g \in G^-$ iff $\alpha = 0$.

The mapping $N : G^+ \to K^*$ (the multiplicative group of K) defined by $N(g) = g^\tau g$, for any $g \in G^+$, is a homomorphism and $N(g)$ is called the spinorial norm of $g \in G^+$. The normal subgroup of G^+ defined as the kernel of N is called the reduced Clifford group and denoted by G_0^+.

The subgroup $\varphi(G_0^+)$ of $SO(q)$ is denoted by $O_0^+(q)$ and called the reduced orthogonal group.

Example:

Let us assume that the ground field K is $\mathbf{R}$, the field of real numbers. Then, $O_0^+(Q)$ coincides with the identity component of the Lorentz group $O(q)$.

1.2.2.6.2 Definitions

The Clifford regular group $\tilde{G}$ is the multiplicative group of invertible elements g in $C(E,q)$ that satisfy, for any $x \in E$, $\pi(g)xg^{-1} = y \in E$.

The linear transformation $\psi(g) : x \to \pi(g)xg^{-1}$ induced by $g \in \tilde{G}$ belongs to the orthogonal group $O(q)$ of E relative to q. The mapping $g \to \psi(g)$ is a homomorphism from $\tilde{G}$ into $O(q)$. Therefore, ψ is a representation of $\tilde{G}$ on E, called the regular vector representationof $\tilde{G}$. The kernel of ψ is K^*. $\tilde{G}$ is identical to the subset of $C(E,q)$ consisting of products of regular (or nonisotropic) vectors of E, and $\tilde{G}$ can be, equivalently, defined as the multiplicative group formed, with the unit element $1_C(E,q)$, by products of regular vectors of E.

If $\tilde{G}^+ = \tilde{G} \cap C^+(E,q)$, $\tilde{G}^- = \tilde{G} \cap C^-(E,q)$, $\tilde{G} = \tilde{G}^+ \bigcup \tilde{G}^-$, we have $\tilde{G}^+ = G^+$, $\tilde{G}^- = G^-$, where $\tilde{G} = G$ if $\dim E$ is even, $\tilde{G}$ is the subgroup $\tilde{G}^+ \bigcup \tilde{G}^-$ of G if $\dim E$ is odd. ψ is φ on $\tilde{G}^+ = G^+$ and ψ is $-\varphi$ on $\tilde{G}^- = G^-$, and $\psi(\tilde{G}^+) = O^+(q)$ and $\psi(\tilde{G}^-) = O^-(q)$.

The mapping $N' : \tilde{G} \to K^*$ defined by $N'(g) = \nu(g).g = g^\nu g = (\pi \circ \tau)g \cdot g$ is a homomorphism from $\tilde{G}$ into the multiplicative group K^* that applies the center $K^* \cdot 1_C$ of $\tilde{G}$ onto $(K^*)^2$. N' is called the graded norm.

For $g \in \tilde{G}$, $g = a_1 \cdots a_k$ with a_j, $1 \leq j \leq k$, regular vectors of E, we have

$$N(g) = \prod_{i=1}^{k} q(a_i)$$

and $N'(g) = (-1)^k N(g)$ and $g^{-1} = g^\tau / N(g) = g^\nu / N'(g)$. N' is N on $\tilde{G}^+$ and N' is $-N$ on $\tilde{G}^-$. The reduced Clifford group $\tilde{G}_0^+ = G_0^+$ appears as the kernel of the homomorphism N or N' from $\tilde{G}^+$ into K^*.

1.2.2.7 The Spin Group Spin (E, q)

We present the following general definition of Spin(E, q).[23]

1.2.2.7.1 Definition For any quadratic regular space (E, q), the spin group is defined to be the normal subgroup of the even Clifford group defined as the kernel of the norm homomorphism[24] according to the following exact sequence:

$$1 \to \mathbf{Z}_2 \to \mathrm{Spin}(E, q) \to O_0^+(q) \to 1.$$

If q is positive, $O_0^+ \simeq SO(n, \mathbf{R})$. Then $G_0^+ = \mathrm{Spin}(E, q)$ is denoted by Spin n and called the classical spinor group of degree n.

We recall the following classical result.[25]

1.2.2.7.2 Proposition (a) *Let (E, q) be a quadratic regular n-dimensional complex space or Euclidean real space. The spin group* Spin(E, q) *is the group consisting of products in the Clifford algebra $C(E, q)$ of an even number of unitary vectors in E.* Spin(E, q) *is connected and simply arcwise connected and constitutes a twofold covering of $SO(E, q)$.*

(b) *Let $(E, q) = E_{r,s}$ be a standard pseudo-Euclidean space of type (r, s),* Spin$(E_{r,s})$ = Spin (r, s), *the corresponding spin group is the group consisting of products in the Clifford algebra $C(E_{r,s})$ of an even number of $a_i \in E$ such that $q(a_i) = 1$ and of an even number of b_j such that $q(b_j) = -1$.*

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[23] Cf., for example, R. Deheuvels, *Tenseurs et Spineurs*, op. cit., pp. 249–255.

[24] In 1.2.2.6.1 such a subgroup was called the reduced Clifford group and denoted by G_0^+.

[25] Cf., for example, R. Deheuvels, *Tenseurs et Spineurs*, op. cit., p. 254.

Spin(r, s) *is connected and simply arcwise connected if* $(r, s) \neq 1$. Spin(r, s) *is a twofold covering of* $SO^+(r, s) = O^{++}(r, s)$*, the identity component of the "generalized Lorentz group* $O(r, s)$*," consisting of proper rotations in* $O(r, s)$ *that preserve the complete orientation of* $E_{r,s}$*. On the other hand,* Spin$(1, 1)$ *has two connected components.*[26]

1.2.2.7.3 Proposition *Let us assume that* $K = \mathbf{R}, \mathbf{C}$*. The Lie algebra* spin$(E, q)$ *of* Spin(E, q) *is the Lie subalgebra of the Lie algebra associated with the associative algebra* $C(E, q)$[27] *consisting of the space* $C_2(E, q)$ *defined above.* spin(E, q) *operates*

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[26] Following Deheuvels (R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit.), we denote by $RO(q)$, for a quadratic regular complex or Euclidean real space, the twofold covering group of $O(q)$ (according to the exact sequence $1 \to \mathbf{Z}_2 \to RO(q) \to O(q) \to 1$); $RO(r, s)$ the twofold covering group of the standard pseudo-Euclidean real space $E_{r,s}$ with $(r, s) \neq (1, 1)$ (according to the exact sequence $1 \to \mathbf{Z}_2 \to RO(r, s) \to O(r, s) \to 1$). We have the following classical exact sequences:

$$1 \to \mathbf{Z}_2 \to \text{Spin}\, n \to SO(n) \to 1,$$
$$1 \to \mathbf{Z}_2 \to \text{Spin}(r, s) \to SO^+(r, s) \to 1 \text{ (with } SO^+(r, s) = O^{++}(r, s)).$$

Some authors, such as Max Karoubi and A. Crumeyrolle and I. Satake, for example, often introduce the following groups: Pin(r, s)—respectively Spin(r, s)—as the subgroup of the regular Clifford group $\tilde{G}$—respectively even Clifford group $\tilde{G}^+$—consisting of elements of $\tilde{G}$, respectively $\tilde{G}^+$, such that $|N'(g)| = |g^\nu g| = 1$. According to their notation, we have

$$1 \to \mathbf{Z}_2 \to \text{Spin}\, n \to SO(n) \to 1,$$
$$1 \to \mathbf{Z}_2 \to \text{Pin}(r, s) \to O(r, s) \to 1, \ (r, s) \neq (1, 1),$$
$$1 \to \mathbf{Z}_2 \to \text{Spin}(r, s) \to SO(r, s) \to 1.$$

(We again clarify the definitions above. When (E, q) is a quadratic regular complex or Euclidean real space, $RO(E, q)$ is the subgroup of the regular Clifford group $\tilde{G}$ consisting of elements $g \in \tilde{G}$ such that $N(g) = 1$ and $RO^+(E, q) = \text{Spin}(E, q)$. If (E, q) is a pseudo-Euclidean standard space $E_{r,s}$, $RO(r, s)$ denotes the group—previously denoted by Pin(r, s)—of elements $g \in \tilde{G}$ such that $N'(g) = \pm 1$ or equivalently $N(g) = \pm 1$. But with our notation Spin(r, s) is the identity component in $RO(r, s)$).

Subsequently, we choose the previous convention according to the following exact sequences:

$$1 \to \mathbf{Z}_2 \to \text{Spin}\, n \to SO(n) \to 1,$$
$$1 \to \mathbf{Z}_2 \to \text{Spin}(r, s) \to O^{++}(r, s) \to 1.$$

In the case $(r, s) = (1, 1)$, E_{11} is a hyperbolic real plane. Each component of $RO(1, 1)$ possesses two arcwise connected components, and thus $RO(1, 1)$ has eight connected components.

[27] It is well known that one can associate with any associative algebra A a Lie algebra by setting $[a \mid b] = ab - ba$ for any $a, b \in A$.

in E by means of the bracket product of $C(E,q)$:

$$\text{If } x \in E, \quad a = \sum_{i<j} \lambda_{ij} e_i e_j \in C_2(E,q), \quad \textit{then } [a,x] = ax - xa \in E.$$

The linear space Spin $(E,q) = C_2(E,q)$ *is isomorphic to the space* $\wedge^2(E)$, *and its dimension over K is* $\frac{1}{2}n(n-1)$. *The linear representation of* Spin(E,q) *onto* $C_2(E,q)$ *by inner automorphisms of* $C(E,q)$ *is naturally the adjoint representation of* Spin(E,q) *in its Lie algebra.*

Classical examples:[28]
Spin 2 $\simeq S^1$;
Spin3 $\simeq SU(2) \simeq SpU(1)$ {group consisting of classical quaternions with norm 1};
Spin 4 $\simeq SU(2) \times SU(2) \simeq S^3 \times S^3$;
Spin 5 $\simeq SpU(2)$;
Spin 6 $\simeq SU(4)$;
Spin$(1,3) \simeq SL(2,\mathbf{C})$ (cf. below, exercises).

1.2.2.8 Spinors and Spin Representations

We recall the following important statement:

1.2.2.8.1 Theorem (structure theorem of Wedderburn[29]) *Any simple algebra[30] is isomorphic to the algebra of endomorphisms of a right vector space M over a field—not necessarily commutative—Γ that is an extension of the ground field K of A. In other words, A is isomorphic to the algebra of square matrices of degree p over the field Γ, where $p = \dim_\Gamma M$. Therefore $\dim_K A = p^2 \dim_K \Gamma$.*

1.2.2.8.2 Structure of Clifford Algebras for Regular Quadratic Spaces

Let E be a K-n-dimensional space. According to Proposition 1.2.2.5.2, the Clifford algebra $C(E,q)$ of a quadratic regular space is a central simple algebra if n is even, the direct sum of two isomorphic central simple algebras both isomorphic to the even Clifford algebra $C^+(E,q)$ if n is odd and if $e_N^2 = (e_1 \cdots e_n)^2 \in (K^*)^2$, and a central simple algebra over the field $\tilde{K} = K(\sqrt{\Delta(q)}) = K(\sqrt{e_N^2})$ if n is odd and if $e_N^2 = \Delta(q) \notin (K^*)^2$. Then the Clifford algebra is always isomorphic to an algebra of square matrices or to the direct sum of two copies of such algebras over a field that is not necessarily commutative.

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[28] We recall the following classical definition: The set of unit vectors $a \in \mathbf{R}^n$ is by definition the unit sphere S^{n-1}.

[29] Cf. for example: R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., p. 340.

[30] We recall that a simple algebra is an algebra A of finite dimension, with a unit element, which has only A and $\{0\}$ for two-sided ideals. A semisimple algebra A an algebra direct sum of a finite number of simple algebras A_i. (Each A_i is a two-sided ideal of A and A is the direct sum of the spaces A_i.)

1.2.2.8.3 Definition Let (E, q) be a quadratic regular space and $C(E, q)$ its Clifford semisimple algebra. By definition we call any minimal faithful module over $C(E, q)$ the space of spinors associated with $C(E, q)$. When S is the direct sum of two simple nonisomorphic modules, $(S = S_1 \oplus S_2)$, $S_i (i = 1, 2)$ are called the spaces of half-spinors.

1.2.2.8.4 Spin-Representations in the Case $K = C(n = \dim E \geq 3)$

In this case $O_0^+ \simeq SO(n, \mathbf{C})$, then we denote G_0^+ by $\mathrm{Spin}(n, \mathbf{C})$ and call it by definition the complex spinor group of degree n.

$\mathrm{Spin}(n, \mathbf{C})$ is a simply connected covering group via the covering homomorphism φ. $\mathrm{Spin}(n, \mathbf{C})$ is the complexification of the compact Lie group $\mathrm{Spin}(n)$ and is a complex analytic subgroup of the complex Lie group $C(E, q)^*$ consisting of all invertible elements of $C(E, q)$.

The spin representations of the group $\mathrm{Spin}(n, \mathbf{C})$[31] are defined as follows:

Case $n = 2r$, (n even)

$C(E, q)$ is a central simple algebra. $C(E, q)$ is isomorphic to a total matrix algebra of degree 2^r over $\mathbf{C}$. $C(E, q)$ possesses, up to an equivalence, a unique irreducible representation ρ of degree 2^r. We call the corresponding space S of this representation *the space of spinors for* $\mathbf{C}$. The representation of C^+, ρ^+ induced by ρ, is called the spin representation of C^+. ρ induces a representation of the Clifford group $\tilde{G}$ of the even Clifford group $\tilde{G}_+ = G^+$, and of the reduced Clifford group $\tilde{G}_0^+$, which are respectively denoted by ρ, ρ^+, and ρ_0^+ and are also called spin representations.

Thus, the restriction ρ^+ of ρ to $\mathrm{Spin}(n, \mathbf{C}) = G_0^+$ (or eventually $\mathrm{Spin}\, n$) defines a representation ρ of degree 2^r of $\mathrm{Spin}(n, \mathbf{C})$ (or eventually $\mathrm{Spin}\, n$). Since C^+ is not simple, ρ^+ is not irreducible: ρ^+ is the sum of two inequivalent simple representations both of degree 2^{r-1}, ρ_+^+, and ρ_-^+, and the same is true for ρ_0^+, the spin representations of $\mathrm{Spin}(n, \mathbf{C})$. Thus, S can be represented in one and only one way as the sum of two subspaces each of which yields a irreducible (or simple) representation: $S = S^+ \oplus S^-$.

By taking a suitable minimal left ideal P of $C(E, q)$ as the representation space of the representation ρ, one obtains the representation of ρ_+^+ by putting $P^+ = P \cap C^+$ and $P^- = P \cap C^-$.

As pointed out by C. Chevalley, we usually choose for ρ the representation $u \in C(E, q) \to \rho(u)$ such that $\rho(u).\ vf = uvf$, where $f = y_1 y_2 \cdots y_r$ with $\{x_i\}_{1 \leq i \leq r}$ and $\{y_j\}_{1 \leq j \leq r}$ two respective bases of the respective maximal totally isotropic subspaces F, F' such that $E = F + F'$ (Witt's decomposition) with, $2B(x_i, y_j) = \delta_{ij}$ and for P the space $C(E, q) f$ with basis $\{x_{i_1} x_{i_2} \cdots x_{i_h} f\}_{1 \leq i_1 < i_2 < \cdots < i_h \leq r}$. Thus $P^+ = C^+ \cap S$ et $P^- = C^- \cap S$. The representations ρ^+ and ρ^- are respectively called the even (or odd) half-spin representations. ρ_{0+}^+ and ρ_{0-}^+ are not well defined on $SO(n, \mathbf{C})$ (or eventually $SO(n)$). They are of valence 2 on these groups. The representations of the Lie algebra $so(n, \mathbf{C})$, a complex Lie algebra of type D_r, corresponding

[31] Cf., for example, C. Chevalley, op. cit., pp. 55–58.

to ρ_{0+}^{+} and ρ_{0-}^{+} are also called half-spin representations of this Lie algebra $so(n, \mathbf{C})$.

Case $n = 2r + 1$ (n odd)

$C(E, q)$ is semisimple and C^+ is central simple. C^+ is isomorphic to a total matrix algebra of degree 2^r over $\mathbf{C}$. C^+ possesses a unique—up to an equivalence—irreducible representation ρ^+ that is of degree 2^r, which we call the spin representation of C^+. The space S of this representation will be called the space of spinors. The induced representations of G^+, G_0^+ induced by ρ^+ are called the spin representations of G^+, respectively G_0^+, and denoted by ρ^+ and ρ_0^+, respectively.

Thus, ρ_0^+ is the spin representation of degree 2^r of $G_0^+ = \mathrm{Spin}(n, \mathbf{C})$ (or eventually of $\mathrm{Spin}\, n$). The corresponding representation of the Lie algebra $so(n, \mathbf{C})$, a complex Lie algebra of type B_r, is also called the spin representation of $so(n, \mathbf{C})$. We note that ρ is not well defined on $SO(n, \mathbf{C})$ (or eventually $SO(n)$); ρ is of valence 2 on $SO(n, \mathbf{C})$ or on $SO(n)$. As pointed out by C. Chevalley, it is possible in exactly two ways to extend the spin representation of C^+ on S to an irreducible representation of C on S. The two representations of C that extend ρ^+ are called the spin representations of C, and the induced representations of G are called the spin representations of G.

1.3 Involutions of Algebras

We recall briefly the main results that can be found in the remarkable book of A. A. Albert.[32]

1.3.1 Classical Definitions

Let A be a unitary algebra over K, K being a commutative field with characteristic different from 2.

1.3.1.1 Definition A nonsingular linear transformation $\mathcal{J}$ over K of A is called a K-involution or briefly an involution of A if $\mathcal{J}^2 = 1$, $(ab)^{\mathcal{J}} = b^{\mathcal{J}} a^{\mathcal{J}}$, for any $a, b \in A$, and whenever such a $\mathcal{J}$ exists, A is called a $\mathcal{J}$-involutorial algebra.

1.3.1.2 Theorem *The product $S = T\mathcal{J}$ of any two involutions of A is an automorphism over K of A.*

1.3.2 $\mathcal{J}$-Symmetric and $\mathcal{J}$-Skew Quantities[33]

1.3.2.1 Definition A quantity s of A is called $\mathcal{J}$-symmetric, respectively $\mathcal{J}$-skew, iff $s = s^{\mathcal{J}}$, respectively $s = -s^{\mathcal{J}}$.

[32] A. A. Albert, *Structure of Algebras*, op. cit. chapter X.

[33] We take the word used by A. A. Albert: "quantity" stands for "element."

1.3.2.2 Theorem *The set $S_{\mathcal{J}}(A)$ of all $\mathcal{J}$-symmetric quantities of A is a linear subset over K of A and is a subalgebra of A iff all $\mathcal{J}$-symmetric quantities are commutative with one another.*

If $A = S_{\mathcal{J}}(A)$, the algebra A is a commutative algebra. Let Z be the center of A. Then $S_{\mathcal{J}}(Z)$ is a subalgebra of Z. The set $\mathcal{C}_{\mathcal{J}}(A)$ of all $\mathcal{J}$-skew quantities of A is a linear subset over K of A.

For any $\mathcal{J}$-involutorial algebra over K, A is the supplementary sum: $A = S_{\mathcal{J}}(A) \oplus \mathcal{C}_{\mathcal{J}}(A)$.

Let A be a $\mathcal{J}$-involutorial algebra over K and let Z be the center of A. If Z contains a regular $\mathcal{J}$-skew quantity $q = -q^{\mathcal{J}}$, then the set $\mathcal{C}_{\mathcal{J}}(A) = qS_{\mathcal{J}}(A)$, q^2 is in $S_{\mathcal{J}}(A)$ and $A = S_{\mathcal{J}}(A) \oplus qS_{\mathcal{J}}(A)$.

1.3.2.3 Definition Let A be $\mathcal{J}$-involutorial over K and Z be the center of A. We call A $\mathcal{J}$-involutorial of the first kind, respectively of second kind, according as $S_{\mathcal{J}}(Z) = Z$ or $S_{\mathcal{J}}(Z) \neq Z$ respectively.

We will now be particularly interested in the case that A is a simple algebra, and we will assume that the center of A is a field $\mathcal{R}$.

Every quantity $q \neq 0$ of a field $\mathcal{R}$ is regular. Moreover, the subset G of all $\mathcal{J}$-symmetric quantities of $\mathcal{R}$ is a subfield over K of $\mathcal{R}$ by Theorem 1.3.2.2. Hence either A is $\mathcal{J}$-involutorial of the first kind or $\mathcal{R}$ contains a quantity q as in Theorem 1.3.2.2.

We have the following theorem (A. A. Albert, op. cit. Theorem 10, p. 153):

1.3.2.4 Theorem *Let the center of a $\mathcal{J}$-involutorial algebra A of the second kind be a field $\mathcal{R}$ such that $\mathcal{R} \supset G$ (G the subfield over K of all J-symmetric quantities of $\mathcal{R}$). Then $\mathcal{R} = G(\theta)$ is a separable quadratic field over G such that*

(i) $\theta^{\mathcal{J}} = 1 - \theta, \theta^2 - \theta = \beta$ *in* G.

(ii) $A = G_{\mathcal{J}}(A) \oplus \theta G_{\mathcal{J}}(A) = \{u_1, \ldots, u_n\}$ *over* $\mathcal{R}$ *with* $u_i = u_i^{\mathcal{J}}$ *in* $G_{\mathcal{J}}(A)$. *Moreover, we can replace θ in (ii) by $q = \theta - 1/2$ and obtain $\mathcal{R} = G(q)$,*

(iii) $q^{\mathcal{J}} = -q, q^2 = \alpha$ *in* G.

1.3.3 Involutions over *G* of a Simple Algebra

We assume henceforth that A is a simple algebra over K. Then the center $\mathcal{R}$ of A is always a field, and the subfield $G_{\mathcal{J}}(\mathcal{R})$ of all $\mathcal{J}$-symmetric quantities of $\mathcal{R}$ is uniquely determined by $\mathcal{J}$. We recall the following definition given before.

1.3.3.1 Definition Let G be a subfield over K of the center $\mathcal{R}$ of a simple algebra A. Then we call an involution $\mathcal{J}$ of A an involution over G of A if $G_{\mathcal{J}}(\mathcal{R}) = G$, that is, $k = k^{\mathcal{J}}$ for k in $\mathcal{R}$ if and only if k is in G.

The result of Theorem 1.3.2.2 now implies that we may limit our study of the existence of involutions over G of a normal simple algebra A over $\mathcal{R}$ to the discussion of the case that $\mathcal{R}$ has an automorphism C over G as follows: Either $\mathcal{R} = G$, C is the identity automorphism I, or $\mathcal{R} = G(\theta)$ is a separable quadratic extension over the set G consisting of all quantities of $\mathcal{R}$ unaltered by C, $C^2 = I$, $\theta^2 - \theta$ is in G,

(iv) $\theta^C = 1 - \theta = \theta^{\mathcal{J}}$.

We will adopt this notation here and henceforth. If T is an involution over G of A, then we have seen that necessarily $k^T = k^C$ for every k of $\mathcal{R}$. If also $\mathcal{J}$ is an involution over G, we have $k^{\mathcal{J}} = k^C = k^T$; $C^2 = I$ gives $k^{T\mathcal{J}} = k$. Combining this result with that of Theorem 1.3.1.2 we have the following result:

1.3.3.2 Lemma *Let T and $\mathcal{J}$ be involutions over G of A over K. Then $T\mathcal{J}$ is an automorphism over the center $\mathcal{R}$ of A.*

Complete information on the relation between any two involutions over the same G of A is now given by the following theorem:

1.3.3.3 Theorem *Let $\mathcal{R}$ be the center of a simple algebra A and T an involution over G of A. Then a self-correspondence $a \to a^{\mathcal{J}}$ is an involution $\mathcal{J}$ over G of A if and only if there exists a regular quantity $y = \pm y^T$ in A such that*

(v) $a^{\mathcal{J}} = y^{-1} a^T y$ *(a in A).*

The correspondence S given by $a \leftrightarrow y^{-1}ay$ is an automorphism of A over $\mathcal{R}$. If $y = \pm y^T$ then the resulting $\mathcal{J}$ of (v) is clearly the product $\mathcal{J} = TS$ and hence is a non-singular linear transformation over G of A. Now $a^{s^{-1}} = yay^{-1}$, $a^{ST} = (y^{-1}ay)^T = y^T a^T (y^T)^{-1} = ya^T y^{-1} = a^{TS^{-1}}$, $ST = TS^{-1}$. Then $\mathcal{J}^2 = TSTS = T^2 S^{-1} S = I$. Also, $(ab)^{\mathcal{J}} = (ab)^{ST} = (a^S b^S)^T = b^{ST} a^{ST}$, so that $\mathcal{J}$ is an involution of A. It is an involution over G since k^T is in $\mathcal{R}$, $k^{TS} = (k^T)^S = k^T$ for every k of $\mathcal{R}$, S is an automorphism over $\mathcal{R}$.

Conversely, let $\mathcal{J}$ be an involution over G. By Lemma 1.3.3.2 and a corollary of the classical theorem of Skolem–Noether,[34] $S = T\mathcal{J}$ is an inner automorphism of A, $a^S = g_0^{-1} a g_0$ for g_0 a regular quantity of A. Then $a^T = a^{S\mathcal{J}} = (g_0^{\mathcal{J}}) a^{\mathcal{J}} (g_0^{\mathcal{J}})^{-1}$, $a^{\mathcal{J}} = g^{-1} a^T g$, where $g = g_0^{\mathcal{J}}$ is regular. We apply $\mathcal{J}$ to $a^{\mathcal{J}} = g^{-1} a^T g$ and obtain $a = g^{\mathcal{J}} (a^T)^{\mathcal{J}} (g^{\mathcal{J}})^{-1} = g^{-1} g^T g g^{-1} a^{T^2} g (g^{-1} g^T g)^{-1} = (g^{-1} g^T) a (g^{-1} g^T)^{-1}$ for every a of A. Then $\gamma = g^{-1} g^T$ is in the center $\mathcal{R}$ of A, $g^T = \gamma g$. If $\gamma = -1$ the

[34] Theorem of Skolem–Noether (A. A. Albert, op. cit. p. 51, for example, or J. P. Serre, Seminaire H. Cartan, E.N.S. 1950–1951, 2^e exposé 7-01: W. A. Benjamin, Inc, 1967, New York, Amsterdam).

Theorem: Let A be a central simple algebra finite over K, and let f and g be two K-isomorphisms from a simple algebra B into A. Then, there exists an invertible element $x \in A$ such that for any $b \in B$, $f(b) = xg(b)x^{-1}$.

Corollary: Any K-automorphism of a central simple algebra finite over K is an inner automorphism.

quantity $y = g = -g^T$ has the property we desire. Otherwise, $y = g + g^T = (1+\gamma)g$ is a T-symmetric quantity and is regular, $y^{-1}a^T y = g^{-1}a^T g = a^{\mathcal{J}}$ as desired.

It is clear, that if $\mathcal{J}$ is any involution over G of A defined by a quantity y of A satisfying (v), the multiples αy of y by nonzero quantities α of $\mathcal{R}$ have the property $(\alpha y)^{-1}a^T(\alpha y) = a^{\mathcal{J}}$ and define the same involution $\mathcal{J}$ of A as y. Conversely, if $a^{\mathcal{J}} = y^{-1}a^T y = y_0^{-1}a^T y_0$ for every a of A, then $y_0 y^{-1} a^T = a^T y_0 y^{-1}$ for every a of A. But $a = (a^T)^T$, $y_0 y^{-1} a = a y_0 y^{-1}$ for every a of A, $y_0 y^{-1} = \alpha$ in $\mathcal{R}$, $y_0 = \alpha y$. We have shown that the quantity y of (v) is uniquely determined by $\mathcal{J}$ up to a nonzero factor in $\mathcal{R}$.

We remark also that if $y^T = y$, then $y^{\mathcal{J}} = y^T = y$, and that if $y^T = -y$, then $y^{\mathcal{J}} = y^T = -y$.

Two involutions $\mathcal{J}$ and $\mathcal{J}_0$ are called *cogredient* if there exists an automorphism S over $\mathcal{R}$ of A such that $\mathcal{J}_0 = S^{-1}\mathcal{J}S$. Then S is an inner automorphism of A over $\mathcal{R}$ and $a^S = z^{-1}az$ for a regular quantity z of A. But then $a^{S^{-1}} = zaz^{-1}$,

$$\text{(vi) } a^{\mathcal{J}_0} = z^{-1}y^{-1}(zaz^{-1})^T yz = (z^T yz)^{-1}a^T(z^T yz) = y_0^{-1}a^T y_0,$$

where $y_0 = z^T yz$. The argument above shows that the two involutions $\mathcal{J}$ and $\mathcal{J}_0$ over G are cogredient if only if the defining y_0 is T-congruent to a multiple of y by a quantity in the center.

The automorphisms S of an algebra may be thought of as replacing any fixed representation a of its abstract arbitrary quantity by another representation a^S. Now $\mathcal{J}_0$ is the involution $a^S \leftrightarrow (a^S)^{\mathcal{J}_0} = a^{SS^{-1}\mathcal{J}S} = (a^{\mathcal{J}})^S$. Thus *cogredient involutions are essentially merely different representations of the same abstract involution.*

1.4 Clifford Algebras for Standard Pseudo-Euclidean Spaces $E_{r,s}$ and Real Projective Associated Quadrics

1.4.1 Clifford Algebras $C_{r,s}$ and $C^+_{r,s}$: A Review of Standard Definitions

Let $V = E_{r,s}$ be the standard m-dimensional pseudo-Euclidean space of type (r, s), $r+s = m$. Let $(x|y) = x^1y^1 + \cdots + x^r y^r - x^{r+1}y^{r+1} - \cdots - x^{r+s}y^{r+s}$ be its scalar product, relative to an orthogonal basis of V, namely $e = \{e_1, \ldots, e_n\}$ with $q(e_i) = 1$, $1 \leq i \leq r$ and $q(e_j) = -1, r+1 \leq j \leq m$. $C(V) = C_{r,s}$ denotes its Clifford algebra. $C_{r,s}$ is an associative algebra with a unit element 1_C, 2^m-dimensional over $\mathbf{R}$. π is its principal automorphism, τ its principal antiautomorphism (main involution of $C_{r,s}$), $\nu = \pi \circ \tau = \tau \circ \pi$ the conjugation in $C_{r,s}$. $C^+_{r,s} = C^+(V)$ denotes its even Clifford subalgebra, 2^{m-1}-dimensional over $\mathbf{R}$. G denotes the regular Clifford groupof $C_{r,s}$. N, respectively N', denotes the usual spinorial norm and the graded norm.

We recall that for $q = x_1 \cdots x_p \in G$, the product of p regular vectors of V, $N'(g) = (-1)^p N(g)$ and $g^{-1} = g^\tau/N(g) = g^\nu/N'(g)$. Let $\mathcal{J} = e_1 \cdots e_m$. If $m = 2k$, $\mathcal{J}^2 = (-1)^{k+s}$, $\mathcal{J}^\tau = \mathcal{J}^\nu = (-1)^k\mathcal{J}$, $\mathcal{J}^{-1} = (-1)^{k+s}\mathcal{J}$. If $m = 2k+1$, $\mathcal{J}^2 = (-1)^{k+s}$, $\mathcal{J}^\tau = (-1)^k\mathcal{J}$, $\mathcal{J}^\nu = (-1)^{k+1}\mathcal{J}$, $\mathcal{J}^{-1} = (-1)^{k+s}\mathcal{J}$.

Spin$V$ = Spin(r, s) is the kernel of the restriction of the homomorphism N to $G^+ = C^+(V) \cap G$. (We recall that G^+ consists of elements that can be written as the products of an even number of regular vectors of V.) For $m > 2$, Spin$V$ is connected and included into the G^{++} subgroup of G^+ consisting of $g \in G^+$ that can be written as the product of an even number of positive vectors of V and of an even number of negative vectors of E.

The group Spin V linearly generates $C^+(V) = C^+(r, s)$—(subalgebra of even elements)—in which it is embedded. Now we are going to study in detail the nature of the algebras $C_{r,s}$ and $C^+_{r,s}$.

1.4.2 Classification of Clifford Algebras $C_{r,s}$ and $C^+_{r,s}$

According to Theorem 1.2.2.5.1, we know that the subalgebra C^+ of C can be expressed as the Clifford algebra of any subspace $E_1 = u^\perp$ of E, orthogonal space to a regular vector u for the quadratic form $q_1 = -q(u)q$.

Thus, let us take a vector u of V such that $(u \mid u) = \varepsilon = \pm 1$. The mapping φ from $u^\perp$ into $C^+(V) : y \in u^\perp \to uy = \varphi(y)$ is such that $(\varphi(y))^2 = -(u \mid u)(y \mid y) = -\varepsilon(y \mid y)$ and represents $C^+(V)$ as the Clifford algebra of the vector space $u^\perp$ endowed with the quadratic form induced from that of V by multiplying by $(-\varepsilon)$ and thus of signature $(r, s-1)$ if $\varepsilon = -1$ or $(s, r-1)$ if $\varepsilon = 1$. All such structures of Clifford algebras for $C^+(V)$ corresponding to different choices of u define the same conjugation, which is identical to the restriction of τ to $C^+(V)$.

One can establish the classifying Table 1.1, which gives explicitly the nature of $C_{r,s}$ and $C^+_{r,s}$, according to $r - s$ modulo 8. Such a result is due to the nature of the Brauer–Wall group:[35] $BW(\mathbf{R}) = \mathbf{Z}/8\mathbf{Z}$. We agree to denote by $m(n, F)$ the real algebra of square matrices of degree n with coefficients in the field $F = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$ (the usual noncommutative field of real quaternions). We denote by $[k]$ the integer part of the real k.

Proof. The construction of such a table entails the knowledge of some properties of periodicity modulo 8 of real quadratic regular spaces.[36]

Let us first recall the following well-known results given in the above reference:
$C_{r,s} \otimes C_{1,1} \simeq C_{r+1,s+1}$ (one can even use a tensor product of $\mathbf{Z}_2$-graded algebras);
$C_{r,s} \otimes C_{0,2} \simeq C_{s,r+2}$; $C_{1,0} \simeq \mathbf{R} \oplus \mathbf{R}$, $C_{0,1} \simeq \mathbf{C}$;
$C_{r,s} \otimes C_{0,2} \simeq C_{s,r+2}$; $C_{2,0} \simeq C_{1,1} \simeq m(2, \mathbf{R})$, $C_{0,2} \simeq \mathbf{H}$;
$m(n, \mathbf{R}) \otimes m(n, \mathbf{R}) \simeq m(nm, \mathbf{R})$; $\mathbf{C} \otimes \mathbf{C} \simeq C \oplus C$;
$m(n, \mathbf{R}) \otimes \mathbf{R} \simeq m(n, \mathbf{R})$;
$m(n, \mathbf{C}) \otimes \mathbf{C} \simeq m(n, \mathbf{C})$;[37]
$m(n, \mathbf{R}) \otimes \mathbf{H} \simeq m(n, \mathbf{H})$; $\mathbf{H} \otimes \mathbf{C} \simeq m(2, \mathbf{C})$, $\mathbf{H} \otimes \mathbf{H} \simeq m(4, \mathbf{R})$;

[35] C. T. C. Wall, Graded algebras anti-involutions, simple groups and symmetric spaces, op. cit.

[36] Cf., for example, T. Y. Lam, *The Algebraic Theory of Quadratic Forms*, op. cit., chapter 5.

[37] Cf. J. P. Serre, Applications algébriques de la cohomologie des groupes, op. cit., p. 603.

Table 1.1. Fundamental Table

$r+s$ (mod 2)	$r-s$ (mod 8)	$C^{+}_{r,s}$	$C_{r,s}$
0	0	$m(2^{[\frac{m-1}{2}]}, \mathbf{R}) \oplus m(2^{[\frac{m-1}{2}]}, \mathbf{R})$	$m(2^{\frac{m}{2}}, \mathbf{R})$
1	1	$m(2^{\frac{m-1}{2}}, \mathbf{R})$	$m(2^{[\frac{m}{2}]}, \mathbf{R}) \oplus m(2^{[\frac{m}{2}]}, \mathbf{R})$
0	2	$m(2^{[\frac{m-1}{2}]}, \mathbf{C})$	$m(2^{\frac{m}{2}}, \mathbf{R})$
1	3	$m(2^{\frac{m-1}{2}-1}, \mathbf{H})$	$m(2^{[\frac{m}{2}]}, \mathbf{C})$
0	4	$m(2^{[\frac{m-1}{2}]-1}, \mathbf{H})$ $\oplus m(2^{[\frac{m-1}{2}]-1}, \mathbf{H})$	$m(2^{\frac{m}{2}-1}, \mathbf{H})$
1	5	$m(2^{\frac{m-1}{2}-1}, \mathbf{H})$	$m(2^{[\frac{m}{2}]-1}, \mathbf{H}) \oplus m(2^{[\frac{m}{2}]-1}, \mathbf{H})$
0	6	$m(2^{[\frac{m-1}{2}]}, \mathbf{C})$	$m(2^{\frac{m}{2}-1}, \mathbf{H})$
1	7	$m(2^{\frac{m-1}{2}}, \mathbf{R})$	$m(2^{[\frac{m}{2}]}, \mathbf{C})$

$C_{0,n+8} \simeq C_{0,n} \otimes C_{0,8}$, $C_{0,8} \simeq m(16, \mathbf{R})$; hence we can deduce that if $C_{0,n} \simeq m(m, F)$, where F is the field $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, we obtain that $C_{0,n+8} \simeq m(16m, F)$, which leads us to the following table first given in Atiyah et al.,[38] and now classical.[39]

n	$C_{n,0}$	$C_{0,n}$	$C^{\mathbf{C}}_n = C_{n,0} \otimes \mathbf{C} \simeq C_{0,n} \otimes \mathbf{C}$
1	$\mathbf{R} \oplus \mathbf{R}$	$\mathbf{C}$	$\mathbf{C} \oplus \mathbf{C}$
2	$m(2, \mathbf{R})$	$\mathbf{H}$	$m(2, \mathbf{C})$
3	$m(2, \mathbf{C})$	$\mathbf{H} \oplus \mathbf{H}$	$m(2, \mathbf{C}) \oplus m(2, \mathbf{C})$
4	$m(2, \mathbf{H})$	$m(2, \mathbf{H})$	$m(4, \mathbf{C})$
5	$m(2, \mathbf{H}) \oplus m(2, \mathbf{H})$	$m(4, \mathbf{C})$	$m(4, \mathbf{C}) \oplus m(4, \mathbf{C})$
6	$m(4, \mathbf{H})$	$m(8, \mathbf{R})$	$m(8, \mathbf{C})$
7	$m(8, \mathbf{C})$	$m(8, \mathbf{R}) \oplus m(8, \mathbf{R})$	$m(8, \mathbf{C}) \oplus m(8, \mathbf{C})$
8	$m(16, \mathbf{R})$	$m(16, \mathbf{R})$	$m(16, \mathbf{C})$

Thus, for example, $C_{14,0} \simeq m(64, \mathbf{H})$ since $14 \equiv 6 \pmod 8$ and $C_{6,0} \simeq m(4, \mathbf{H})$. Since $C_{r,s} \otimes C_{1,1} \simeq C_{r+1,s+1}$, if we assume that $r > s$, we obtain

$$C_{r-s,0} \otimes \underbrace{C_{1,1} \otimes \cdots \otimes C_{1,1}}_{s \text{ factors}},$$

[38] M. F. Atiyah, R. Bott, and A. Shapiro, *Clifford Modules*, op. cit., p. 12.
[39] Cf., for example, D. Husemoller, *Fibre Bundles*, op. cit., p. 161.

and furthermore $C_{1,1} \simeq m(2, \mathbf{R})$[40], whence we deduce that $C_{r,s} \simeq C_{r-s,0} \otimes m(2^s, \mathbf{R})$ and if $C_{r-s,0}$ is isomorphic to $m(m, F)$, we find that $C_{r,s} \simeq m(m, F) \otimes m(2^s m, F)$, which leads us to a previous case.

If we assume that $r < s$, the study of the nature of $C_{r,s}$ leads to that of $C_{0,s-r}$, whence we can deduce the nature of $C_{r,s}$ in a similar way. As for the nature of $C^+_{r,s}$, it is sufficient to recall that according as a fundamental remark (1.2.2.e), $C^+_{r,s}$, can be realized as the Clifford algebra $C(E_1)$, with $E_1 = u^\perp$, with $(u \mid u) = \varepsilon = \pm 1$ and therefore endowed with a quadratic form of signature $(r, s-1)$ if $\varepsilon = -1$ or $(s, r-1)$ if $\varepsilon = 1$.

For example, the even Clifford algebra $C^+_{4,7}$ is isomorphic to $m(2^4, \mathbf{H})$ as $4 - 7 \equiv -3 \equiv 5 \pmod 8$, and the Clifford algebra $C_{4,7}$ is isomorphic to $m(2^4, \mathbf{H}) \oplus m(2^4, \mathbf{H})$.

1.4.3 Real Projective Quadrics $\tilde{Q}(E_{r,s})$

We recall some classical results that will be developed in every detail in Chapter 2.

Let us consider again the standard pseudo-Euclidean regular space of type (r, s), $V = E_{r,s}$, with $m = r + s = \dim V$, with its standard scalar product $(x|y) = x^1y^1 + \cdots + x^r y^r - x^{r+1}y^{r+1} - \cdots - x^{r+s}y^{r+s}$ in an orthogonal basis $e = \{e_1, \ldots, e_m\}$ with $q(e_i) = 1$ for $1 \leq i \leq r$ and $q(e_j) = -1$ for $r+1 \leq j \leq r+s$.

1.4.3.1 Definition The isotropic cone Q, minus its origin, is a differentiable singular submanifold of $V = E_{r,s}$. If P denotes the projection from $V \setminus \{0\}$ onto its associated projective space $P(V)$, $\tilde{Q} = P(Q \setminus \{0\})$ is naturally provided with a pseudo-Riemannian conformal structure of type $(r-1, s-1)$. $\tilde{Q} = \tilde{Q}(E_{r,s})$ is called, by definition, the standard real projective quadric of type (r, s).

1.4.3.2 Theorem (Definition) *Let $F = V \oplus H$, where H is the standard real hyperbolic plane equipped with an isotropic basis $\{\varepsilon, \eta\}$ such that $2(\varepsilon, \eta) = 1$. Therefore, F is a standard regular pseudo-Euclidean vector space of type $(r+1, s+1)$. Let $Q(F)$, $(m+1)$-dimensional, denote its isotropic cone. $M = P(Q(F) \setminus \{0\})$, image into $P(F)$ of the isotropic cone minus its origin of F, is m-dimensional and called the compactification of $V = E_{r,s}$. M is identical to the homogeneous space $PO(F)/\mathrm{Sim}(V)$,[41] quotient group of $PO(F) = O(r+1, s+1)/\mathbf{Z}_2$ by the group $\mathrm{Sim}\,V$ of similarities of V.*

[40] Directly for C_{11}, there are four basis elements $1, e_1, e_2$, and e_1e_2 with $e_1^2 = 1, e_2^2 = -1, e_1e_2 = -e_2e_1, (e_1e_2)^2 = 1, (e_1e_2)e_1 = -e_2, (e_1e_2)e_2 = -e_1$. If we map $1 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, e_1 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_2 \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and then $e_1e_2 \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we get an algebra isomorphism between C_{11} and $m(2, \mathbf{R})$.

[41] It will be shown in Chapter 2 that in fact $PO(F) = O(r+1, s+1)/\mathbf{Z}_2$ can be called the conformal group of $V = E_{r,s}$.

Then, we have the following statement:

1.4.3.3 Proposition *There is a natural mapping from $S^r \times S^s$ onto the projective quadric $M = \tilde{Q}(F)$ that leads to the identification of M with the quotient of the manifold $S^r \times S^s$ by the equivalence relation $(a, b) \sim (-a, -b)$, and thus $S^r \times S^s$ becomes a twofold covering space of M, connected if r and s both are different from zero.*

If r and s are both ≥ 2, $S^r \times S^s$ is simply connected and is the universal covering space of M, the fundamental group of which is $\mathbf{Z}_2$.

If r or $s = 1$, $S^r \times S^s$ is not simply connected and the fundamental group of M is infinite. The special case of $s = 0$ is studied below. Let $F = V \oplus H$. Let $\{e_1, \ldots, e_r, e_{r+1}, \ldots, e_{r+s}\}$ be the standard orthonormal basis of V and $\{e_0, e_{n+1}\}$ be a basis of H such that for any x in H, $x = x^0 e_0 + x^{n+1} e_{n+1}$, $(x|x) = (x^0)^2 - (x^{n+1})^2$.

The equation of the cone $Q(F)$ is the following

$$x = (x^0, x^1, \ldots, x^{n+1}) \in Q(F) \quad \textit{if and only if} \quad \sum_{i=0}^{r} (x^i)^2 - \sum_{l=r+1}^{n+1} (x^l)^2 = 0.$$

The Euclidean sphere of radius $\sqrt{2}$ associated with the basis $\{e_0, \ldots, e_{n+1}\}$ of F has the following equation:

$$\sum_{i=0}^{r} (x^i)^2 + \sum_{l=r+1}^{n+1} (x^l)^2 = 2.$$

x belongs to the intersection of $Q(F)$ and of the sphere if and only if

$$\sum_{i=0}^{r} (x^i)^2 = \sum_{j=r+1}^{n+1} (x^j)^2 = 1,$$

that is, if and only if x belongs to the product of the unit sphere S^r of the standard Euclidean space E_{r+1}, with the basis $\{e_0, \ldots, e_r\}$, by the unit sphere S^s of the standard Euclidean space E_{s+1}, with the basis $\{e_{r+1}, \ldots, e_{n+1}\}$. Let y be any point belonging to $Q(F) \setminus \{0\}$. Necessarily $\sum_{j=0}^{r+1} (y_i)^2 = \sum_{j=r+1}^{n+1} (y^j)^2 = l > 0$. The generator line to which y belongs cuts $S^r \times S^s$ at the two points $\pm \frac{1}{\sqrt{l}} y$. Conversely any couple of points $(a, b) \in S^r \times S^s$ belongs to generator line of $Q(F)$, which it determines.

We have found a natural mapping from $S^r \times S^s$ onto the projective quadric $M = \mathbf{P}(Q(F) \setminus \{0\})$ which leads to the identification of M with the quotient of the manifold $S^r \times S^s$ by the equivalence relation: $(a, b) \sim (-a, -b)$. Therefore $S^r \times S^s$ becomes a two–fold covering of M, and is connected if r and s are both different from zero. If r or s is equal to zero, M is not simply connected.

Let a_1 and a_2 be two orthogonal vectors of S^r, and b_1 and b_2 two other orthogonal vectors of S^s. Then, for any $\theta \in \mathbf{R}$, the point $x_\theta = (a_1 \cos(\theta) + a_2 \sin(\theta), b_1 \cos(\theta) + b_2 \sin(\theta))$ belongs to $S^r \times S^s$.

When θ continuously describes the segment $[0, \pi]$, x_θ describes, in $S^r \times S^s$, a couple of half big-circles which join (a_1, b_1) to $(-a_1, -b_1)$ and its image by the projection $\mathbf{P}$ in M describes a continuous closed path $(\{x_\theta\})$, with origin and endpoint $\mathbf{P}((a_1, b_1))$, which cannot be continuously deformed into a point, with keeping fixed its origin and its endpoint. $\mathbf{P}(\{x_\theta\})$, which is the image in $\mathbf{P}(F)$ of the plane $\{(a_1, b_1), (a_2, b_2)\}$, is a projective line belonging to M, and any generator line of M is of such a type.

If r and s are more than 2, $S^r \times S^s$ is simply connected and is the universal covering of M, the fundamental group of which is $\mathbf{Z}_2$ (any line in M is a "generator" of the group).

If r or $s = 1$, $S^r \times S^s$ is not simply connected and the fundamental group of M is infinite.

Suppose $s = 0$. The equation of the cone $Q(F)$ is then $(x^0)^2 + \cdots + (x^n)^2 - (x^{n+1})^2 = 0$, and that of S^n is $(x^0)^2 + \cdots + (x^n)^2 = 1$. Let $\tilde{\pi}$ be the mapping from the projective quadric M onto S^n defined by

$$\tilde{x} = (x^0, \ldots, x^{n+1}) \in M \mapsto \tilde{\pi} x = \left(\frac{x^0}{x^{n+1}}, \ldots, \frac{x^{n+1}}{x^{n+1}} \right) \in S^n.$$

The restriction to $S^n \times \{1\}$ of the projection from $\mathbf{Q}(F) \setminus \{0\}$ onto M and the mapping $\tilde{\pi}$ are inverse to each other.

The group $\tilde{\pi} \circ PO(n+1, 1) \circ \tilde{\pi}^{-1}$ is called by definition the Möbius group of the sphere, in agreement with the introduction to Chapter 2. It is classically the group of conformal isometries of S^n onto S^n for $n \geq 2$, and is classically generated by inversions of E_{n+1}, which leave globally invariant S^n, and orthogonal symmetries of E_{n+1}. The conformal group of S^n is strictly larger than its subgroup of isometries; the difference of their dimensions is $\frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2} = n + 1$. This property is specific for the spheres. In fact, any compact Riemannian manifold, whose conformal group is strictly larger than its subgroup of isometries is necessarily isometric to a sphere (cf. below Chapter 2).

In a recent paper,[42] Arkadiusz Jadczyk has found another way to study the conformal group of the sphere S^n. His method uses transformers of Gilbert and Murray and the properties of the trace in Clifford algebras to construct a two-fold covering group $\mathrm{Spin}^+(1, n+1)$—called also Spoin by other authors—of the conformal group of the sphere S^n. His results are in complete agreement with the results given below in Chapter 2.

[42] A. Jadczyk, Quantum Fractals on n–spheres. Clifford Algebra Approach, Advances in Applied Clifford Algebra, Vol. 17 no 2, December 2006.

1.5 Pseudoquaternionic Structures on the Space S of Spinors for $C^+_{r,s}$, $m = 2k+1$, $r - s \equiv \pm 3 \pmod 8$. Embedding of Corresponding Spin Groups $\mathrm{Spin} E_{r,s}$ and Real Projective Quadrics $\tilde{Q}(E_{r,s})$[43]

1.5.1 Quaternionic Structures on Right Vector Spaces over H

1.5.1.1 Structure of the Principal Automorphism of H

Let **H** be the usual **R**-associative algebra of real quaternions with "units": $1, i, j, k$ (cf. 1.2.1 above). Let ν denote the usual conjugation defined for $q = \alpha + i\beta + j\gamma + k\delta \in \mathbf{H}$ by $q^\nu = \alpha - i\beta - j\gamma - k\delta$. **H** can be identified with the Clifford algebra $C_{0,2}$ (cf. 1.4.2 above).

According to the general Theorem 1.2.2.5.1 above, we know that **H** is a central simple algebra over **R** with center **R**.

Furthermore, we can apply the fundamental Theorem 1.3.3.3 concerning involutions of simple central algebras to **H**. Then, any involution α of **H** is the composite of the conjugation ν and of an inner automorphism directed by an element u, determined up to a nonzero factor in **R**, which is either ν-symmetric or ν-skew: $q^\alpha = u^{-1} q^\nu u$ with $u^\nu = u$ or $u^\nu = -u$. If $u^\nu = u$, we have $u^\alpha = u^\nu = u$. If $u^\nu = -u$, then $u^\alpha = u^\nu = -u$.

Moreover, the principal automorphism π of the real Clifford algebra **H** is naturally the mapping $q \to k^{-1} q k = -kqk$ since $k^{-1} = -k$, according to Theorem 1.2.2.5.3 above. $\mathbf{H} = \frac{(-1,-1)}{\mathbf{R}}$ and if e_1, e_2 are the elements of the orthogonal basis of $E_{0,2}$ such that $e_1^2 = e_2^2 = -1$, the four "units" of **H** are 1, $e_1 = i$, $e_2 = j$, and $e_1 e_2 = k$, and $k = e_1 e_2$ belongs to the anticenter of $C_{0,2}$. Thus π is an inner automorphism "directed" by k.

1.5.1.2 The Groups $SpU(p, q)$ and $SO^*(2n)$

1.5.1.2.1 Algebraic Remark

Let us take E, a right vector space over **H** with $\dim_{\mathbf{H}} E = n$, with basis $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_n\}$.

Let b be a sesquilinear form on E.[44]

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[43] All the main results of the following sections were given in the following paper: P. Anglès, Algèbres de Clifford $C_{r,s}$ des espaces quadratiques pseudo-euclidiens standards $E_{r,s}$ et structures correspondantes sur les espaces de spineurs associés. Plongements naturels des quadriques projectives $Q(E_{r,s})$ associés aux espaces $E_{r,s}$, op. cit.

[44] We recall briefly that b is a mapping from $E \times E$ into **H** such that for any $x, y, x_i, y_i \in E$ $(i = 1, 2)$, for any $q \in \mathbf{H}$, $b(x, yq) = b(x, y)q$; $b(xq, y) = q^\nu b(x, y)$; $b(x_1 + x_2, y) = b(x_1, y) + b(x_2, y)$; $b(x, y_1 + y_2) = b(x, y_1) + b(x, y_2)$. $b(x, y) = ({}^t X)^\nu B Y$, where B, X, Y are the respective matrices relative to ε for b, x, and y.

By restriction of the noncommutative field **H** to **C**, E naturally becomes a **C**-vector space with basis $\tilde{\varepsilon} = \{\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_1 j, \ldots, \varepsilon_n j\}$, and an easy computation shows that the components of b, with respect to the complex structure of E, are respectively the complex forms h and a defined for any $x, y \in E$ by $b(x, y) = h(x, y) + ja(x, y)$, where a is a complex bilinear form and h is a sesquilinear form, *linear in the second argument and antilinear in the first one*.

By using the fundamental Theorem 1.3.3.3 *we can lead the study of* **H***-skew sesquilinear forms on* E *back to that of* **H***-symmetric ones by changing the involution of* **H**.

If b is **H**-skew for ν, i.e., for any $x, y \in E$, $b(y, x) = -(b(x, y))^\nu$ let us put $g(x, y) = b(y, x)k$,[45] id est $b(x, y) = g(x, y)k^{-1}$. Then,

$$\begin{aligned} g(y, x) &= b(y, x)k = -((b(x, y))^\nu)k = -(g(x, y)k^{-1})^\nu k = (g(x, y)k)^\nu k \\ &= k^\nu (g(x, y))^\nu k = -kg(x, y)^\nu k = \pi(g(x, y))^\nu = (\pi \circ \nu)(g(x, y)) \\ &= (g(x, y))^\tau \end{aligned}$$

since $\pi \circ \nu = \tau$ and according to 1.5.1.1. Then g is **H**-symmetric for the involution τ of **H**.

1.5.1.2.2 The Group $SpU(p, q)$

Let E be a right n-dimensional vector space over **H**.

1.5.1.2.2.1 Definition A sesquilinear **H**-symmetric form denoted by $\{\,|\,\}$ on E such that for any $x \in E - \{0\}$, $\{x \mid x\} > 0$, is called a quaternionic scalar product on E. An easy computation shows that the components of $\{\,|\,\}$ are respectively a hermitian scalar product denoted by $\langle\,|\,\rangle$ and a symplectic scalar product denoted by $[\,|\,]$. Therefore, for any $x, y \in E$, $\{\mathbf{x} \mid \mathbf{y}\} = \langle \mathbf{x} \mid \mathbf{y}\rangle + \mathbf{j}[\mathbf{x} \mid \mathbf{y}]$.

If $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_n\}$ is the standard basis of E over **H**, which is orthonormal for the quaternionic scalar product, let us put

$$x = \sum_{i=1}^{n} \varepsilon_i x^i \text{ and } y = \sum_{i=1}^{n} \varepsilon_i y^i.$$

We put $x^i = \xi^i + j\xi^{n+i}$ and $y^i = \eta^i + j\eta^{n+i}$. We observe that for any $z \in$ **C**, $jz = \bar{z}j$, where for $z = \alpha + i\beta$, $\bar{z} = \alpha - i\beta$ is the classical conjugate of z. Then

$$\{x|y\} = \sum_{i=1}^{n} (x^i)^\nu y^i = \sum_{k=1}^{2n} \bar{\xi}^k \eta^k + j \sum_{i=1}^{n} (\xi^i \eta^{n+i} - \xi^{n+i} \eta^i).$$

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b is **H**-symmetric (or **H**-hermitian) iff for any $x, y \in E$ we have $b(y, x) = b(x, y)^\nu$ and thus for any $x \in E$, $b(x, x) \in$ **R**, the field of real numbers. b is **H**-skew (or **H**-anti-hermitian) iff $b(y, x) = -b(x, y)^\nu$.

[45] k is the classical "unit element" of **H**.

Thus we have the following theorem:

1.5.1.2.2.2 Definition (Theorem[46]) The symplectic unitary group of degree n is, by definition, the group consisting of automorphisms of a quaternionian right space E over $\mathbf{H}$, that leave invariant the quaternonian scalar product of E, the group law being the classical composition. We have $SpU(E) = U(E, \mathbf{C}) \cap Sp(E, \mathbf{C})$.

Example:

Let H^n be the classical standard n-dimensional right space over $\mathbf{H}$. We put

$$\{x \mid y\} = \sum_{i=1}^{n} (x^i)^\nu y^i, \text{ where } x = \sum_{i=1}^{n} \varepsilon_i x^i \text{ and } y = \sum_{i=1}^{n} \varepsilon_i y^i, \ \varepsilon = \{\varepsilon_1, \dots, \varepsilon_n\}$$

being the standard canonical basis.

Then we find that $SpU(\mathbf{n}) = \mathbf{U(2n)} \cap \mathbf{Sp(2n, C)}$, where $U(2n)$ is the standard unitary group of $\mathbf{C}^{2n}$ and $Sp(2n, \mathbf{C})$ the standard symplectic group on $\mathbf{C}^{2n}$. If we assume that E is provided with a pseudoquaternionian scalar product of type (p, q), which can be written in an orthogonal basis $\varepsilon = \{\varepsilon_1, \dots, \varepsilon_n\}$ as

$$\{x|y\} = \sum_{i=1}^{p} (x^i)^\nu y^i - \sum_{i=p+1}^{p+q} (x^i)^\nu y^i$$

with $p+q = n$, we find again, by the same method that the pseudoquaternionic group of type (p, q) associated with the pseudoquaternionic scalar product of type (p, q), namely $SpU(p, q)$ appears as

$$SpU(p, q) = U(2p, 2q) \cap Sp(2(p+q), \mathbf{C}).$$

Since classically, $U(2p, q) = SO(2p, 2q) \cap Sp[2(p+q), \mathbf{R}]$, we obtain the following result:

1.5.1.2.2.3 Proposition $SpU(p, q) = SO(4p, 4q) \cap Sp(2(p+q), \mathbf{C}) \cap Sp(4(p+q), \mathbf{R})$.

Thus, E becomes a vector space over $\mathbf{R}$, the field of real numbers, of dimension $4n = 4(p+q)$ over $\mathbf{R}$, a basis of which over $\mathbf{R}$ is

$$\{\varepsilon_1, \dots, \varepsilon_n, \varepsilon_1 i, \dots, \varepsilon_n i, \varepsilon_1 j, \dots, \varepsilon_n j, \varepsilon_1 k, \dots, \varepsilon_n k\}.$$

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[46] Another proof is the following:

$$\begin{aligned} u \in SpU(E) &\Leftrightarrow \forall\, x, y \in E,\ \{u(x) \mid u(y)\} = \{x \mid y\} \\ &\Leftrightarrow \begin{cases} \forall\, x, y \in E,\ \langle u(x) \mid u(y)\rangle = \langle x \mid y\rangle \\ \text{and } [u(x) \mid u(y)] = [x \mid y] \end{cases} \\ &\Leftrightarrow SpU(E) = U(E, \mathbf{C}) \cap Sp(E, \mathbf{C}). \end{aligned}$$

Let us consider the antilinear operator of the complex $2n$-dimensional space E defined for any $x \in E$ by $T(x) = xj$. (The antilinearity comes from the fact that for any $x \in E$, $(xi)j = -(xj)i$.) We have clearly $T^2 = -I$. Conversely, the datum of such an operator T on a complex vector space E implies that the dimension of E is even[47] and allows us to define a quaternionian structure by putting

$$x \cdot q = x(z + jz') = xz + (xj)z' = xz + T(x) \cdot z'.$$

We can now formulate the following statement.

1.5.1.2.2.4 Theorem *$SpU(p, q)$ is the set of elements $u \in U(2p, 2q)$ such that $u \circ T = T \circ u$. $SpU(p, q)$ is the set of elements $u \in Sp(2(p + q), \mathbf{C})$ such that $u \circ T = T \circ u$.*

For this purpose, let us write first the fact that $h \in U(2p, 2q)$ and that $u \circ T = T \circ u$. Then, $h(u(x), u(y)) = h(x, y)$ for any $x, y \in E$ is equivalent to $u \in U(2p, 2q)$. If $u \circ T = T \circ u$, since we have classically for any x, y; $h(xj, y) = a(x, y)$, then $a(u(x), u(y)) = h(u(x)j, u(y)) = h(T \circ u(x), u(y)) = h(u \circ T(x), u(y))$, which implies that $a(u(x), u(y)) = h(T(x), y)$ since $u \in U(2p, 2q)$ and therefore

$$a(u(x), u(y)) = h(T(x), y) = h(xj, y) = a(x, y) \text{ and } u \in Sp(2(p + q), \mathbf{C}).$$

Conversely, let us assume that $u \in Sp(2(p + q), \mathbf{C})$ and that $u \circ T = T \circ u$. Then, $a(u(x), u(y)) = a(x, y)$ for any $x, y \in E$. Thus, classically,[48] $h(u(x), u(y)) = -a(u(x)j, u(y)) = -a(T \circ u(x), u(y)) = -a(u \circ T(x), u(y)) = -a(T(x), y)$ since $u \in Sp(2(p + q), \mathbf{C})$. Thus we obtain that $h(u(x), u(y)) = -a(T(x), y) = h(x, y)$ and then $u \in U(2p, 2q)$.

1.5.1.2.3 The Group $SO^*(2n)$ as a Quaternionic Group

Let us take b an $\mathbf{H}$-skew sesquilinear form on E such that $b(x, y) = h(x, y) + ja(x, y)$, where h is a $\mathbf{C}$-skew hermitian form on the $2n$-dimensional complex space E, and a is a symmetric bilinear complex form on E. According to 1.1.5.2, we know that there exists an orthonormal basis $\{\varepsilon_l\}_{1 \le l \le n}$ of E such that $b(\varepsilon_l, \varepsilon_l) = j$ ("unit" quaternion with $j^2 = -1$). Therefore we can deduce according to another result of J. Dieudonné[49] that all nondegenerate $\mathbf{H}$ skew-hermitian on E are equivalent with maximal index $[\frac{n}{2}]$, where n is the dimension of E over $\mathbf{H}$ ($[r]$ is the integer part of

[47] The proof is easy and the same used for the following classical result: The study of complex n-dimensional spaces E is identical to the study of real finite-dimensional spaces F provided with a linear operator $\mathcal{J}$ such that $\mathcal{J}^2 = -I$. If E denotes a complex n-dimensional space, ${}_{\mathbf{R}}E$, the real associated space obtained by restriction of C to $\mathbf{R}$, we have that E is identical to $({}_{\mathbf{R}}E, \mathcal{J})$.

[48] Cf. R. Deheuvels, Formes quadratiques et groupes classiques, op. cit., p. 441.

[49] J. Dieudonné, On the structure of unitary groups, op. cit., p. 383.

the real number r). We put

$$x = \sum_{l=1}^{n} \varepsilon_l x^l, \quad y = \sum_{l=1}^{n} \varepsilon_l y^l$$

with $x^l = \xi^l + j\xi^{n+l}$, $y^l = \eta^l + j\eta^{n+l}$ with ξ^l and η^l belonging to $\mathbf{C}$. We obtain that

$$b(x, y) = \sum_{l=1}^{n} (\bar{\xi}^l - j\eta^{n+l}) j(\eta^l + \xi j^{n+l}) = \sum_{l=1}^{n} (-\bar{\xi}^l \eta^{n+l} + \bar{\xi}^{n+l}\eta^l) + j \sum_{l=1}^{n} \xi^l \eta^l.$$

1.5.1.2.3.1 Definition (Theorem) The unitary group of automorphisms of E that leave b invariant is denoted by $U_n(E, b)$, which we agree to call, by definition, the *symplectoquaternionic group* $U_n(E, b) = U_{2n}(h, \mathbf{C}) \cap O(2n, \mathbf{C})$, *where* $U_{2n}(h, \mathbf{C})$ *denotes the unitary group for the skew-hermitian complex form* E.

According to a previous definition given in 1.1.3.2, the special unitary corresponding group $SU_n(E, b)$, the group consisting of elements of $U_n(E, b)$ with determinant equal 1, is $SU_n(E, b) = SU_{2n}(h, \mathbf{C}) \cap SO(2n, \mathbf{C})$, and can be identified with $SO^*(2n)$, $SO^*(2n) = SU_n(E, b)$.

1.5.2 Invariant Scalar Products on Spaces S of Spinors

We want to present a general method initiated by R. Deheuvels[50] for the special case $m = r + s = 4k + 2$, $r - s = 4l + 2$ following another general idea of André Weil.[51] The Ariadne's thread is the following one for the case that *the ground field K is commutative.*

1.5.2.1 Definition Let E be vector space over a commutative field K. By definition we call a scalar product on E any K-symmetric or K-skew nondegenerate bilinear form b on E.

As already said (1.1.5.2), with any K-linear operator u belonging to the algebra $\mathcal{L}_K(E)$ of linear operators of E we can associate its adjoint operator u^* defined by $b(ux, y) = b(x, u^*y)$ for any x, y in E. *The adjunction*, with respect to b, $* : u \to u^*$ *is an involution of* $\mathcal{L}_K(E)$.

An easy computation shows that we also have $b(x, uy) = b(u^*x, y)$. Moreover, if $\lambda \in K^*$, the mapping $(x, y) \to \lambda b(x, y)$ is another "scalar product" that determines the same adjunction as b on $\mathcal{L}_K(E)$, with the same invariance group and the same linear subspaces of symmetric operators u (such that $u = u^*$) or skew operators (such that $u = -u^*$) (cf. exercises).

Conversely, one can verify that if we assume that b and b', both scalar products on E, have the same adjunction on $\mathcal{L}_K(E)$, then there exists $\lambda \in K^*$ such that $b'(x, y) = \lambda b(x, y)$ for any x, y in E.

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[50] R. Deheuvels, (a) Groupes conformes et algèbres de Clifford, Rend. Sem. Mat. Univer. Politecn. Torino, vol 43, 2, 1985, pp. 205–226. (b) Tenseurs et spineurs. P.U.F. Paris, 1993.

[51] A. Weil, Algebras with involutions and the classical groups, Collected papers, vol. II, pp. 413–447, Springer-Verlag, New York, 1980.

Furthermore, since the field of complex numbers **C** often plays an important role, let us consider E a vector space over a commutative field L provided with an involutive automorphism $\mathcal{J} : \lambda \to \bar{\lambda} = \mathcal{J}(\lambda)$.[52] Let K be the subfield of elements μ of K such that $\bar{\mu} = \mu$ and let be $q \in L$ be such that $\mathcal{J}(q) = \bar{q} = -q \neq 0$. Then the subset K_- of elements $z \in L$ such that $\bar{z} = -\mathcal{J}(\lambda)$ is a vector subspace of the K-vector space L, and we have $L = K \oplus K_- = K \oplus qK$ (compare with Theorems 1.3.2.2 and 1.3.2.4 for involutions of algebras).

1.5.2.2 Definition By definition a hermitian scalar product respectively skew-hermitian—relative to $\mathcal{J}$ on E—is any sesquilinear nondegenerate form b on E that satisfies $b(y, x) = \varepsilon b(x, y)^{\mathcal{J}}$ with $\varepsilon = 1$ in the case of a hermitian form and $\varepsilon = -1$ in the case of a skew-hermitian form. The corresponding adjunction $* : u \to u^*$ is an involution of the algebra $\mathcal{L}_K(E) : (uv)^* = v^*u^*, u^{**} = u, (\lambda u)^* = \bar{\lambda}u^* = \lambda^{\mathcal{J}}u^*$.[53]

Then we have the following result:

1.5.2.2.1 Proposition *If b is a hermitian scalar product on E, then qb is a skew-hermitian scalar product on E, where q is defined above in 1.5.2.1, so that $\bar{q} = -q \neq 0$. If b is a skew-hermitian scalar product on E, then qb is a hermitian scalar product on E. Both b and qb have the same corresponding adjunction. The proof is immediate and is left as a simple exercise.*

1.5.2.2.2 Theorem *Let E be a space over the field K, let $(\,|\,)$ denote any nondegenerate scalar product on E, and let * be the corresponding adjunction defined on $\mathcal{L}_K(E)$. Let u be any invertible element in $\mathcal{L}_K(E)$. The following two statements are equivalent to each other*

(i) *The inner automorphism $a \mapsto u^{-1}au$ of $\mathcal{L}_K(E)$ commutes with *.*
(ii) *u is a similarity of E relative to $(\,|\,)$, i.e., there exists $\lambda \in K^*$ such that $(ux|uy) = \lambda\,(x|y)$ for any $x, y \in E$.*

Note: Classically the statement (ii) is equivalent to (ii)': u preserves the orthogonality id est for any $x, y \in E$ if $(x|y) = 0$, then $(ux|uy) = 0$. (cf. exercise (V) at the end of this chapter). To prove the above theorem, let θ_u be the inner automorphism $a \mapsto u^{-1}au$ of $\mathcal{L}_k(E)$. We put $u = v^{-1}$. Then θ_u commutes with * if and only if $(vav^{-1})^* = v^{-1*}a^*v^* = va^*v^{-1}$. Then, setting $\beta = va^*v^{-1} = \alpha = v^{-1*}a^*v^*$, we compute $v^*va^* = v^*\beta v = v^*\alpha v = v^*v^{-1*}a^*v^*v = a^*v^*v$. Thus we find that $v^*va^* = a^*v^*v$. Conversely, if $v^*va^* = a^*v^*v$, then, with $\beta_1 = a^*v^*v$ and $\alpha_1 = v^*va^*$, we have $v^{-1*}a^*v^* = v^{-1*}\beta_1 v^{-1} = v^{-1*}\alpha_1 v^{-1} = v^{-1*}v^*va^*v^{-1} = va^*v^{-1}$, which explains the fact that θ_u commutes with *.

Thus θ_u commutes with * if and only if $v^*va^* = a^*v^*v$ id est if and only if $v^*v = \lambda I$, $\lambda \in K^*$, where I denotes the identity element of $\mathcal{L}_K(E)$ or, equivalently if and only if $u^*u = \mu I$, $\mu \in K^*$. Moreover, by definition of the adjunction *, we have that $(vx|vy) = (x|v^*vy)$ for any $x, y \in E$. Therefore (i) implies (ii) since if

[52] Since the field L is commutative, $\mathcal{J}$ is also an *involution* (antiautomorphism) of L.

[53] Cf. 1.1.5.2

$v^*v = \lambda I$ — that is equivalent to (i) —, then we have (ii)': If $v^*v = \lambda I$, $(vx|vy) = \lambda(x|y)$ and $(x|y) = 0$ implies that $(vx|vy) = 0$ and therefore $(ux|uy) = 0$. Conversely, since $(\,|\,)$ is nondegenerate, if, by assumption, $(ux|uy) = \mu(x|y)$ and on the other hand $(ux|uy) = (x|u^*uy)$ we find that we have: $(x|\lambda y - u^*uy) = 0$ and therefore $u^*u = \mu I$ with $\mu \in K^*$.

We know that all the Clifford algebras have two fundamental involutions: the principal antiautomorphism τ and the conjugation $\nu = \pi \circ \tau = \tau \circ \pi$. We can now ask the following problem:

Do there exist scalar products on the space of spinors for which τ and ν are possible adjunctions? (One can show that such scalar products are unique up to a nonzero scalar.)

We are going to study in detail the case $m = r + s = 2k + 1, r - s \equiv \pm 3 \pmod 8$.

1.5.3 Involutions on the Real Algebra $L_{\mathbf{H}}(S)$ where S is a Quaternionic Right Vector Space on H, with $\dim_{\mathbf{H}} S = n$

1.5.3.1 Introductory Notes

Let S be a quaternionic right space over **H** with $\dim_{\mathbf{H}} S = n$. According to (1.5.1.2), we know that by changing the involution of **H**, if b is an **H** skew-hermitian form for ν the classical conjugation of **H**, then b is an **H**-hermitian form for $\tau = \pi \circ \nu$ the principal antiautomorphism of the Clifford algebra $H = \frac{(-1,-1)}{R}$. *Thus, we are led to the study of* **H***-hermitian scalar products or pseudoquaternionic scalar products* on S. With such a general pseudoquaternionic scalar product b on S, we can associate the adjunction $a \to a^*$ in the real algebra $\mathcal{A} = L_{\mathbf{H}}(S)$ such that $(a+b)^* = a^* + b^*$, $(a^*)^* = a$, $(ab)^* = b^*a^*$, $(\lambda.1)^* = \lambda.1$, for any $\lambda \in \mathbf{R}$ and for any $a, b \in \mathcal{A}$.

The real algebra $m(n, \mathbf{H})$—isomorphic to $L_{\mathbf{H}}(S)$—of square matrices of degree n on the field **H** is provided with the structure of a right quaternionic space of dimension n^2 on **H**. A suitable basis for such a structure consists of the n^2 matrices ε_{ij}, $1 \le i, j \le n$, such that the only nonzero coefficient of the matrix ε_{ij} is that of the row i and column j, which is 1. The adjoint of A is then $A^* = {}^t A^\nu$, where A^ν, is the conjugate of A.

We are now going to show that any involution α of the real algebra $L_{\mathbf{H}}(S)$ can be considered as the adjunction for a nondegenerate sesquilinear form on S.

If $\mathcal{A}$ is a central simple algebra over a commutative field K according to Wedderburn's theorem,[54] $\mathcal{A}$ is isomorphic to $L_\Gamma(S)$, where S is a right vector space on the

[54] We present again the result already given (1.2.2.8.1)

Wedderburn's theorem: *Let A be a simple algebra with a unit element over a commutative field K, of finite dimension over K. Then A is isomorphic to an algebra of matrices on a not necessarily commutative field Γ, extension of K, that contains K in its center and is finite over K.*

Thus, any simple algebra A, with a unit element, is isomorphic to the algebra of endomorphisms of a right vector space M over a not necessarily commutative field Γ, extension of K, which means that A is isomorphic to the algebra of all square matrices of degree p ($p = \dim_\Gamma M$) with coefficients in Γ. Then $\dim_K A = p^2 \dim_K \Gamma$. Cf., for example, field Γ, a not necessarily commutative finite extension of K. Since there $K = \mathbf{R}$, according to a general classical result of Weierstrass and Frobenius,[55] $\Gamma = \mathbf{C}$ or $\Gamma = \mathbf{H}$.

According to the fundamental table of 1.4.2 (case $r - s \equiv \pm 3 \pmod 8$), for the study of corresponding $C^+_{r,s}$, the corresponding field with which we are concerned is therefore that where $\Gamma = \mathbf{H}$, the noncommutative field of quaternions.

According to previous remarks (1.2.2.8), $\mathcal{A}$ possesses a simple $\mathcal{A}$-module and can be identified[56] with the real algebra of linear operators of the right quaternionic space S.

Let $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_n\}$ be an arbitrary basis of S. Such a basis determines on S a quaternionic standard scalar product[57] and ε is an orthonormal basis for this standard quaternionic scalar product. Any element a in $\mathcal{A}$ is represented by its matrix A relative to the basis ε, and the adjunction $*$ is such that $A^* =^t A^\nu$. According to the fundamental Theorem 1.3.3.3, if α is an involution of $\mathcal{A}$ we have for any matrix A associated with a with obvious notations, $A^\alpha = U^{-1}({}^t A^\nu)U$ with ${}^t U^\nu = U$ or ${}^t U^\nu = -U$.

Moreover, if ${}^t U^\nu = U$, then U is the matrix, in the basis ε, of a nondegenerate H-symmetric sesquilinear form that determines on S a pseudoquaternionic scalar product, the adjunction of which is precisely α.

If ${}^t U^\nu = -U$, then U is the matrix, in the basis ε, of a nondegenerate H-skew sesquilinear form on S of maximal index $[\frac{n}{2}]$, according to (1.5.1.2), the adjunction of which is precisely α.

Furthermore, we know that in such a case, by changing the involution of H, we are led to the first case.

Now we are going to study the following problem:

If the involution α on the central simple algebra $\mathcal{A} = L_H(S)$ is associated with a pseudoquaternionic scalar product of signature (p, q), determine the signature of such a pseudoquaternionic scalar product for which α is precisely the adjunction.

1.5.3.2 Associated Form with an Involution α on $\mathcal{A} = L_H(S)$

$\mathcal{A} = L_{\mathbf{H}}(S)$ is a real central simple algebra. Since $\dim_{\mathbf{H}} S = n$, and $\dim_{\mathbf{R}} \mathbf{H} = 4$, according to Wedderburn's theorem (1.5.2.8.1) $\dim_{\mathbf{R}} \mathcal{A} = 4n^2$. As in a paper of André

J. P. Serre: Seminaire H. Cartan. E.N.S., 1950–1951, exposé 6-01, W. A. Benjamin Inc. 1967, New York, Amsterdam.

55 Cf., for example, Marcel Riesz, Lecture Series, Clifford numbers and spinors: Lectures delivered, October 1957, January 1958, the institute for fluid dynamics and applied mathematics, University of Maryland, 1957–1958, p. 21. The result of Weierstrass and Frobenius is more general: All finite associative division algebras over the real field are isomorphic to either the real or complex field or to the quaternion algebra.

56 For more precisions; cf. R. Deheuvels, Formes quadratiques et groupes classiques, op. cit., chapitre VIII.

57 Such a scalar product $\{\,|\,\}$ is such that $\{\varepsilon_i \mid \varepsilon_j\} = \delta_{ij}$, $1 \le i, j \le n$, and thus for any

$$x, y \in S, \{x \mid y\} = \sum_{i=1}^{n} (x^i)^\nu y^i \text{ with } x = \sum_{i=1}^{n} \varepsilon_i x^i \text{ and } y = \sum_{i=1}^{n} \varepsilon_i y^i.$$

Weil,[58] let us consider $l(a)$ the endomorphism $x \to ax$ of the underlying vector space to A. Then $\mathrm{Tr}(l(a))$ is well defined as the trace of the regular representation of $\mathcal{A}$.

The trace $\mathrm{Tr}(l(a))$ is invariant under all automorphisms of $\mathcal{A}$, and since $\mathcal{A}$ is semisimple, $\mathrm{Tr}(l(a))$ is also invariant under all antiautomorphisms of $\mathcal{A}$. If λ is such an antiautomorphism of $\mathcal{A}$, $\mathrm{Tr}(l(x^\lambda y))$ is a nondegenerate bilinear form on $\mathcal{A} \times \mathcal{A}$. Since $(x^\lambda y)^\lambda = y^\lambda x^{\lambda^2}$, and therefore $\mathrm{Tr}(l(x^\lambda y)) = \mathrm{Tr}(l(y^\lambda x^{\lambda^2})$, the bilinear form $(x, y) \to \mathrm{Tr}(l(x^\lambda y))$ is symmetric if and only if $\lambda^2 = 1$, i.e., if and only if λ is an involution of $\mathcal{A}$. Thus, we are led to the following definition:

1.5.3.2.1 Definition Let α be an involution of $\mathcal{A} = L_{\mathbf{H}}(S)$. The mapping $(x, y) \to \mathrm{Tr}(l(x^\alpha y))$ is a nondegenerate symmetric bilinear form on $\mathcal{A}$ called the form associated with the involution α.

1.5.3.3 Signature of the Quadratic Form $x \to \mathrm{Tr}(l(x^\alpha x))$

$m(n, \mathbf{H})$ is a right vector space of dimension n^2 over $\mathbf{H}$ with a standard basis consisting of the n^2 matrices ε_{ul} (cf. above 1.5.3.1) $(1 \le u, t \le n)$ over $\mathbf{H}$ and a basis ε' over $\mathbf{R}$ consisting of the $4n^2$ elements $\{\varepsilon_{ut}, \varepsilon_{ut} i, \varepsilon_{ut} j, \varepsilon_{ut} k\}$ (cf. 1.5.1.2.2.3).

Let us assume that there exists on S a pseudoquaternionic scalar product of type (p, q), the adjunction of which is α. We want to determine the signature of the quadratic real form, defined on $\mathcal{A} = L_{\mathbf{H}}(S)$ by $x \to \mathrm{Tr}(l(x^\alpha x))$.[59]

We have just seen (1.5.3.1) that there exists on S a pseudoquaternionic scalar product of signature (p, q) the adjunction of which is precisely α. Let us take for ε an orthogonal basis for such a scalar product. One can verify immediately that the corresponding basis ε' is an orthogonal basis for the bilinear symmetric real form

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[58] A. Weil, op. cit., p. 601.

[59] Let us add some supplementary remarks. Let $m(n, \mathbf{H})$ be provided with its structure of a ring and of a quaternionic right vector space of dimension n^2 over $\mathbf{H}$. Let $l(A) : B \in m(n, \mathbf{H}) \to AB$; $l(A)$ is a linear mapping from $m(n, \mathbf{H})$ into $m(n, \mathbf{H})$. As pointed out by J. Dieudonné (J. Dieudonné, Les determinants sur un corps non commutatif, *Bull. Soc. Math. de France*, 71, 1943, pp. 27–45), one can define $\mathrm{Tr}(l(A)) = n \,\mathrm{Tr}\, A \in \mathbf{H}$. Moreover, $Tr({}^t A^\nu) = (Tr\ A)^\nu$. Then $\mathrm{Tr}(l({}^t A^\nu)) = (\mathrm{Tr}(l(A)))^\nu$.

Let us take α, an involution of $\mathcal{A}$ such that for any $a, b \in \mathcal{A}$, for any $\lambda \in \mathbf{R}$, $(ab)^\alpha = b^\alpha a^\alpha$, $(a^\alpha)^\alpha = a$, $(\lambda.1)^\alpha = \lambda.1$, for any $\lambda \in \mathbf{R}$. The translation of these facts in $m(n, \mathbf{H})$ is the following: $(AB)^\alpha = B^\alpha A^\alpha$, $(A^\alpha)^\alpha = A$, $(\lambda.Id)^\alpha = \lambda.Id$ for any $A, B \in m(n, \mathbf{H})$ and for any $\lambda \in \mathbf{R}$.

In other words, α is an $\mathbf{R}$-linear mapping from $m(n, \mathbf{H})$ into $m(n, \mathbf{H})$ and α is an antiautomorphism for the ring structure of $m(n, \mathbf{H})$. One can easily verify that $\mathrm{Tr}(l^t (A^\alpha)^\nu) = \mathrm{Tr}(l(A))$. As pointed out by A. Weil (op. cit, p. 601), if A is semisimple the right-hand and left-hand regular representations are equivalent, and then the trace is invariant under all *antiautomorphisms* of A. Then consider the mapping $A \to^t (A^\alpha)^\nu$. Therefore, $\mathrm{Tr}(l(A^\alpha)) = (\mathrm{Tr}(l(A)))^\nu$. Thus the mapping f from $m(n, \mathbf{H}) \times m(n, \mathbf{H})$ into $\mathbf{H}$ defined by $f(A, B) = Tr(l(A^\alpha B))$ is such that $f(B, A) = (f(A, B))^\nu$, $f(A + A', B) = f(A, B) + f(A', B)$, $f(A, B + B') = f(A, B) + f(A, B')$ for any A, B, A', B' in $m(n, \mathbf{H})$ and $f(\lambda A, B) = \lambda f(A, B) = f(A, \lambda B)$ for any $\lambda \in \mathbf{R}$.

$(x, y) \to \mathrm{Tr}(l(x^\alpha y))$. Let $H = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$ be the matrix of the pseudoquaternionic scalar product relative to ε, with $\alpha_1, \ldots, \alpha_p > 0$ and $\alpha_{p+1}, \ldots, \alpha_{p+q} < 0$ with $p + q = n$. $A^\alpha = H^{-1}({}^t A^\nu)H$. If

$$A = \sum_{i,j} \varepsilon_{ij} a_{ij},$$

we find that

$$\mathrm{Tr}(l(A^\alpha A)) = \sum_{i,j} \frac{\lambda_i}{\lambda_j} |a_{ij}|^2,$$

where $|a_{kl}| = (a_{kl}^\nu a_{kl})^{1/2}$ is the classical absolute value of the quaternion a_{kl} (for $a_{kl} = \alpha + i\beta + j\gamma + k\delta$, $|a_{kl}|^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$). Then we have obtained the following statement:

1.5.3.1 Theorem *The signature of the quadratic real form defined on* $L_{\mathbf{H}}(S)$ *by* $A \to \mathrm{Tr}(l(A^\alpha A))$ *is* $(4(p^2 + q^2) + 8pq)$. (*We observe that* $4(p^2 + q^2) + 8pq = 4n^2 = \dim_{\mathbf{R}} L_{\mathbf{H}}(S)$.)

1.5.4 Quaternionic Structures on the Space S of Spinors for $C^+_{r,s}$, $r + s = m = 2k + 1$, $r - s \equiv \pm 3 \pmod 8$

We know how to get a realization of $C^+_{r,s}$, by the choice of a vector $u \in E_{r,s}$ such that $(u|u) = \epsilon = \pm 1$: we consider the Clifford algebra $C(E_1)$ with $E_1 = u^\perp$, E_1 the standard space of type $(r, s-1)$ if $\varepsilon = -1$ or $(s, r-1)$ if $\varepsilon = 1$.

According to the fundamental theorem (1.3.3.3), any involution α of $\mathcal{A} = C^+_{r,s}$ can be written with notation in terms of matrices: $A^\alpha = U^{-1}({}^t A^\nu)U$ with $({}^t U^\nu) = U$ or $({}^t U^\nu) = -U$. If $({}^t U^\nu) = U$, then $U^\alpha = ({}^t U^\nu) = U$, and if $({}^t U^\nu) = -U$, then $U^\alpha = ({}^t U^\nu) = -U$.

We can now specify the element U that is determined up to a nonzero scalar factor. According to Theorem 1.2.2.5.1, if we realize $C^+_{r,s}$ as $C(E_1) = C(u^\perp)$ such that $(u \mid u) = -1$, we can take $\mathcal{J}_1 \in C^+_{r,s}$ to be the product of the elements of the basis of E_1 chosen, and U is proportional to $\mathcal{J}_1$.

If we realize $C^+_{r,s}$ as $C(u^\perp)$ such that $(u \mid u) = 1$, we can take $\mathcal{J}'_1$ to be the product of the elements of the basis of E_1 chosen, and U is proportional to $\mathcal{J}'_1$. We obtain the following table:

Realization of $C^+_{r,s}$:

$C(u^\perp)$, $(u\|u) = -1$	$C(u^\perp)$, $(u\|u) = 1$
$u^\perp$ of type $(r, s-1)$	$u^\perp$ of type $(s, r-1)$
$\mathcal{J}_1^2 = (-1)^{k+s-1}$	$\mathcal{J}_1'^2 = (-1)^{k+s}$
$\mathcal{J}_1^\tau = (-1)^k \mathcal{J}_1$	$\mathcal{J}_1'^\tau = (-1)^k \mathcal{J}'_1$

$m = r + s = 2k + 1, r - s \equiv 3\,(8), k \equiv s + 1\,(4)$

$\mathcal{J}_1^2 = 1$	$\mathcal{J}_1'^2 = -1$
s even (and r odd)	
$\mathcal{J}_1^\tau = -\mathcal{J}_1, N(\mathcal{J}_1) = -1$	$\mathcal{J}_1'^\tau = -\mathcal{J}_1', N(\mathcal{J}_1') = 1$
$\mathcal{J}_1 \notin \mathrm{Spin}(E_{r,s})$	$\mathcal{J}_1' \in \mathrm{Spin}(E_{r,s})$
s odd (and r even)	
$\mathcal{J}_1^\tau = \mathcal{J}_1, N(\mathcal{J}_1) = 1$	$\mathcal{J}_1'^\tau = \mathcal{J}_1', N(\mathcal{J}_1') = -1$
$\mathcal{J}_1 \in \mathrm{Spin}(E_{r,s})$	$\mathcal{J}_1' \notin \mathrm{Spin}(E_{r,s})$

$m = r + s = 2k + 1, r - s \equiv -3\,(8), k \equiv s + 2\,(4)$

$\mathcal{J}_1^2 = -1$	$\mathcal{J}_1'^2 = 1$
s even (and r odd)	
$\mathcal{J}_1^\tau = \mathcal{J}_1, N(\mathcal{J}_1) = -1$	$\mathcal{J}_1'^\tau = \mathcal{J}_1', N(\mathcal{J}_1') = 1$
$\mathcal{J}_1 \notin \mathrm{Spin}(E_{r,s})$	$\mathcal{J}_1' \in \mathrm{Spin}(E_{r,s})$
s odd (and r even)	
$\mathcal{J}_1^\tau = -\mathcal{J}_1, N(\mathcal{J}_1) = 1$	$\mathcal{J}_1'^\tau = -\mathcal{J}_1', N(\mathcal{J}_1') = -1$
$\mathcal{J}_1 \in \mathrm{Spin}(E_{r,s})$	$\mathcal{J}_1' \notin \mathrm{Spin}(E_{r,s})$

The cases k even and k odd appear naturally. Let us take for the involution α the principal antiautomorphism τ of $C_{r,s}^+$. $\mathcal{J}_1^\tau = \mathcal{J}_1$ (respectively $\mathcal{J}_1'^\tau = \mathcal{J}_1'$) if and only if k is even, and $\mathcal{J}_1^\tau = -\mathcal{J}_1$ (respectively $\mathcal{J}_1'^\tau = -\mathcal{J}_1'$) if and only if k is odd.

1.5.4.1 Case That k Is Even

Then $C_{r,s}^+$ is a central simple real algebra isomorphic to $m^2(2^{\frac{m-1}{2}-1}, \mathbf{H})$ according to the fundamental table (1.4.2), and the minimal module of $C_{r,s}^+$ is the space S of spinors associated with $C_{r,s}^+$. (We recall that $C_{r,s}^+$ can be identified with the real algebra $L_{\mathbf{H}}(S)$, with $\dim_{\mathbf{H}} S = 2^{k-1}$.) According to Wedderburn's theorem, $\dim_{\mathbf{R}} C_{r,s}^+ = 2^{2k} = (\dim_{\mathbf{H}} S)^2 \times \dim_{\mathbf{R}} \mathbf{H} = (2^{k-1})^2 \times 4$.

Furthermore, we have seen that with any involution α of $C_{r,s}^+ \simeq L_{\mathbf{H}}(S)$ we can associate the bilinear symmetric real form $(x, y) \to \mathrm{Tr}(l(x^\alpha y))$. Since $V = E_{r,s}$ is a quadratic regular space of type (r, s), let us take an orthogonal basis of V: $\{e_1, \ldots, e_m\}$. We know that the 2^{2k} elements $e_{i_1} \cdots e_{i_{2k}}$ with $1 \leq i_1 < i_2 < \cdots < i_{2k} \leq 2k$ constitute a basis of $C_{r,s}^+$.

If we take $e_I = e_{i_1} \cdots e_{i_{2k}}$ and $e_L = e_{j_1} \cdots e_{j_{2L}}$ both elements of this basis of $C_{r,s}^+$, $e_I^\tau.e_L$ is also an element of this basis and is a nonscalar element if $I \neq L$. In such a case the translation $l(e_I^\tau e_L)$ is a permutation of the elements of this basis without any fixed element and with trace zero.

Moreover, such a basis of $C_{r,s}^+$ is also an orthogonal basis for the above bilinear symmetric real form associated with τ. $e_I^\tau.e_I = N(e_I) = (e_{i_1} \mid e_{i_1}) \cdots (e_{i_{2k}} \mid e_{i_{2k}})$. Then $\mathrm{Tr}(l(e_I^\tau.e_I)) = 2^{2k} N(e_I)$ and $\mathrm{Tr}(l(e_I^\tau.e_I))$ is positive if and only if e_I contains an even number of negative vectors of the basis of V and therefore an even number of positive vectors of this basis.

Moreover, if $p(m)$, respectively $i(m)$ denotes the number of subsets of the set $\{1, \ldots, m\}$ that have even cardinality, respectively odd cardinality, we have $p(m) = i(m) = 2^{m-1}$. Let us assume that $C^+_{r,s}$ is realized as $C(E_1)$ with $E_1 = u^\perp$, where $(u \mid u) = \varepsilon = \pm 1$. If $\varepsilon = 1$, $E_1 = E_{s,r-1}$ and if $\varepsilon_1 = -1$, $E_1 = E_{r,s-1}$. We put $E_1 = E_{m_1,m_2}$. The number of positive vectors of the basis of $C^+_{r,s} = C(E_1) = C(E_{m_1,m_2})$ is $p(m_1)p(m_2)$, and the number of negative vectors of this basis is $i(m_1)i(m_2)$. Since $p(m_1)p(m_2) = i(m_1)i(m_2)$, the quadratic form $x \to \mathrm{Tr}(l(x^\tau x))$ is a neutral form.

Since we have seen that the signature of this quadratic form is $(4(p^2+q^2), 8pq)$, we find that $p^2+q^2 = 2pq$, i.e., $p = q = \frac{1}{2}\dim_{\mathbf{H}} S = 2^{k-2} = 2^{\frac{m-1}{2}-2}$. The pseudoquaternionic scalar product is also a neutral one. Furthermore, the pseudounitary symplectic group of automorphisms of S that leave invariant this pseudoquaternionic scalar product consists of elements u of $L_{\mathbf{H}}(S) \simeq C^+_{r,s}$ such that $u^\tau u = 1$. We have obtained the following theorem:

1.5.4.1.1 Theorem *The space of spinors S associated with the Clifford algebras $C^+_{r,s}$ ($r+s = 2k+1$, k even and $r-s \equiv \pm 3$ (mod 8)) is provided with a natural pseudoquaternionic structure and a pseudoquaternionic neutral scalar product, determined up to a nonzero scalar factor, invariant under the spin group* $\mathrm{Spin}\, V = \mathrm{Spin}\, E_{r,s}$. *For $m \geq 7$ we have the following embedding:* $\mathrm{Spin}\, E_{r,s} \subseteq SpU\big(2^{\frac{m-1}{2}-2}, 2^{\frac{m-1}{2}-2}\big)$.

Since $SpU\big(2^{\frac{m-1}{2}-2}, 2^{\frac{m-1}{2}-2}\big)$ is embedded into $U\big(2^{\frac{m-1}{2}-1}, 2^{\frac{m-1}{2}-1}\big)$ (cf. 1.5.1.2.2.2 above), we are led to prove that in fact, $\mathrm{Spin}\, E_{r,s}$ is embedded in $SU\big(2^{\frac{m-1}{2}-1}, 2^{\frac{m-1}{2}-1}\big)$, i.e., that all the elements of $\mathrm{Spin}\, E_{r,s}$ have determinant = 1, as linear operators of S. We give a general demonstration that can be applied in any case.

Any element g of $\mathrm{Spin}\, E_{r,s}$ can be written as $g = u_1 u_2 \cdots u_{2h}$, a product of an even number of vectors u_i in $E_{r,s}$ with $N(u_i) = (u_i \mid u_i) = 1$ and of an even number of vectors u_j in $E_{r,s}$ with $N(u_j) = (u_j \mid u_j) = -1$. Since $u_1 u_2 = u_2(u_2^{-1} u_1 u_2)$ and classically $y_1 = u_2^{-1} u_1 u_2 \in E_{r,s}$ with $N(y_1) = N(u_1)$, we can always assume that the elements u_i with $N(u_i) = -1$, if they exist, are taken first in the writing of g.

Moreover, if two such elements u_i are linearly dependent, by using such permutations as above, we are led to a factor ± 1. Then, we can assume that $g = u_1 \cdots u_{2h}$ with u_i two by two linearly independent, with u_i such that $N(u_i) = -1$ are taken first, if they exist. If u_i satisfy $u_i^2 = 1 = N(u_i)$, u_i is an involutive linear operator of S with determinant $= \pm 1$. Then, let u_1 and u_2 be two consecutive linearly independent vectors with $N(u_1) = N(u_2) = -1$ and let P be the plane that they generate. Now, there exists, if $r \geq 2$, $z \in E_{r,s}$ such that $(z \mid z) = 1$, $(z \mid u_1) = (z \mid u_2) = 0$ and $(zu_1)^2 = 1$, $(zu_2)^2 = 1$, $zu_1 zu_2 = -u_1 u_2$. Thus zu_1 and zu_2 are involutive linear operators of S with a determinant $= \pm 1$ (cf. Appendix).

Therefore, any element g of $\mathrm{Spin}\, E_{r,s}$ is the product of elements that have determinant $= \pm 1$. $\mathrm{Spin}\, E_{r,s}$ is thus contained into the subgroup of the unitary group $U\big(2^{\frac{m-1}{2}-1}, 2^{\frac{m-1}{2}-1}\big)$ consisting of elements ± 1. But since $\mathrm{Spin}\, E_{r,s}$ is connected ($m > 2$), all these elements have necessarily determinant 1.

We have obtained the following result:

1.5.4.1.2 Theorem *For* $m \geq 7$, $r+s = 2k+1$, k *even,* $r-s \equiv \pm 3$ *(mod 8),* Spin $E_{r,s} \subseteq SU\big(2^{\frac{m-1}{2}-1}, 2^{\frac{m-1}{2}-1}\big)$.

1.5.4.2 Case That *k* Is Odd

In this case, the space S of spinors for $C^+_{r,s}$ is provided with a nondegenerate $\mathbf{H}$-skew sesquilinear form b of index $\frac{1}{2}\dim_{\mathbf{H}} S = 2^{\frac{m-1}{2}-2}$. The group of automorphisms of S that leave b invariant consists of elements u of $C^+_{r,s} \simeq L_{\mathbf{H}}(S)$ such that $u^\tau u = 1$. Spin $E_{r,s}$ is included in this group. The same demonstration as above leads to the following conclusion: Spin $E_{r,s} \subseteq SO^*(2p)$, where $p = 2^{\frac{m-1}{2}-2}$, by showing that all the elements of Spin $E_{r,s}$ have as linear operators of S determinant 1. Moreover, $SO^*(2p)$ is naturally included in $SU(p,p)$, as pointed out by I. Satake.[60] We have thus obtained the following result:

1.5.4.2.1 Theorem *The space of spinors S for Clifford algebras* $C^+_{r,s}$, $r+s = 2k+1$, $r-s \equiv \pm 3$ *(mod 8),* k *even, is provided with a nondegenerate* $\mathbf{H}$*-skew sesquilinear form. Moreover,* Spin $E_{r,s} \subseteq SO^*(2^{\frac{m-1}{2}}) \subseteq SU(2^{\frac{m-1}{2}-1}, 2^{\frac{m-1}{2}-1})$ *for* $m \geq 7$.

With $r+s = 2k+1$, one can verify that one can always find a realization of $C^+_{r,s}$ such that for $r-s = 3\,(\text{mod } 8)$, r and k odd, and s even, there exists an operator $\mathcal{J}'_1 \in \text{Spin}(E_{r,s})$ of S such that $\mathcal{J}'^2_1 = -1$. Similarly, for $r-s = -3\,(\text{mod } 8)$, k and s odd, and r even, there exists $\mathcal{J}_1$ with the same properties. The above justifies, in each case, the structure obtained: SO^*.

1.5.5 Embedding of Projective Quadrics

1.5.5.1 Review of General Results[61]

Let us take again the space S_0 of spinors for the standard Clifford algebra $C(E,q)$ of a quadratic regular space (E,q). Let $(\,|\,)$ denote the associated scalar product. Any vector x in E is represented by a linear operator of S_0. If x is an isotropic vector, we have $x^2 = 0$ inside $C(E,q)$, and then $\text{Im}\, x \subset \text{Ker}\, x$.

If x is isotropic and different from zero, as classically, one can find y, an isotropic vector such as $2(x|y) = 1$ or, equivalently, in $C(E,q)$ such as $xy + yx = 1$. We notice that

(i) $(xy)^2 = (1-yx)xy$ and $(yx)^2 = yx$,

(ii) $(xy)(yx) = 0 = (yx)(xy)$.

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60 Ichiro Satake: Algebraic structures of symmetric domains, op. cit., p. 278.

61 The method has been initiated for the case $r-s \equiv 2$ (mod 4), r and s even, which is equivalent to $m = r+s = 4k+2$ and $r-s = 4l+2$, by R. Deheuvels, Rend. Sem. Mat. Univer. Politecn. Torino, vol. 43, 2, 1985, pp. 205–226.

The two supplementary idempotents (xy) and (yx) depend only on the hyperbolic oriented plane $H = \{x, y\}$. Any orthogonal symmetry of H interchanges them. They both belong to $C^+(E, q)$ and are represented in any space on which $C^+(E, q)$ operates, in S_1 for example, by two supplementary projections.

Let x' and y' be two other isotropic vectors such that $2(x'|y') = 1$, which generate the hyperbolic plane H' assumed different from H. At least one of the vectors x, y, for example x, does not belong to H'. The two decompositions of the unit element associated with H and H', $1 = xy + yx = x'y' + y'x'$, are therefore different. If $xy = x'y'$ we have $xx'y' = 0$ in $C(E, q)$ and if $xy = y'x'$ we have $xy'x' = 0$ in $C(E, q)$.

Classically, we know that the product of three linearly independent vectors of E is always different from zero in $C(E, q)$. Moreover, in S_1, if $s \in \operatorname{Ker} x$, we have $s = (xy+yx)s = (xy)s \in \operatorname{Im} x$. Therefore we obtain that $\operatorname{Im} x = \operatorname{Ker} x = \operatorname{Im}(xy) = \operatorname{Ker}(xy)$ is a linear subspace of dimension $\frac{1}{2}\dim S_1$.

Let us now consider the space S of spinors for $C^+(E, q)$, which is a subspace (proper or not) of S_1, invariant under the action of $C^+(E, q)$. Let us denote by $(xy)_S$, respectively $(yx)_S$, the respective projections of S defined by the respective elements (xy) and (yx) in $C^+(E, q)$. Then, we put $S(x) = \operatorname{Im}(xy)_S = \operatorname{Ker}((yx)_S) = (\operatorname{Im} x) \cap S = (\operatorname{Ker} x) \cap S$. For any $\lambda \neq 0$, $S(\lambda x) = S(x)$.

Let $\{\,|\,\}$ be a scalar product on S associated with the involution τ, i.e., such that for any $a \in C^+(E, q)$ the linear operators that represent respectively a and a^τ in S are adjoint to each other with respect to $\{\,|\,\}$. We have $(xy)^\tau = (yx)$ and $(yx)^\tau = (xy)$. Therefore, for any s, t in S, $\{(x, y)s \mid (xy)t\} = \{s \mid (yx)(xy)t\} = 0$. The subspace $S(x) = \operatorname{Im}(xy)_S$ is totally isotropic for $\{\,|\,\}$. Since $S = \operatorname{Im}(xy)_S \oplus \operatorname{Im}(yx)_S$ is a direct sum, $S = S(x) \oplus S(y)$ is a direct sum of two totally isotropic subspaces; then these subspaces are maximal totally isotropic both of dimension $\frac{1}{2}\dim S$. Such a demonstration shows that the scalar product $\{\,|\,\}$ is necessarily neutral if there exist isotropic nonzero vectors in E.

If x and x' are two isotropic vectors of E that generate a regular plane, the results above show that we have $S = S(x) \oplus S(y)$ and then $S(x) \cap S(y) = 0$, and moreover, $S(x)$ is different from $S(y)$. Furthermore, if x and x' are two linearly independent isotropic vectors that generate a totally isotropic plane, there exist classically[62] two isotropic vectors y and y' that generate a totally isotropic plane orthogonal to $\{x, x'\}$ such that $xy + yx = 1 = x'y' + y'x'$, while $xy' + y'x = x'y + yx' = 0 = xx' + x'x = yy' + y'y$. We put $p = (xy)_S$, $q = (yx)_S$, $p' = (x'y')_S$, $q' = (y'x')_S$. We can now deduce the following obvious results: $pq' = q'p$, $p'q = qp'$, $pp' = p'p$, $qq' = q'q$. If we assume $S(x) = S(x')$, i.e., $\operatorname{Im} p = \operatorname{Im} p'$, then $q'p = 0$ and $qp' = 0$, whence $p = p(p' + q') = pp' = p'p = p'$ and then $q = q'$, and since $C^+(E, q) \subset \mathcal{L}(S)$, $xy = x'y'$ and $yx = y'x'$, a contradictory result. Thus we have obtained the following:

1.5.5.1.1 Theorem *The mapping*

$$\{\textit{isotropic line } Kx \textit{ of } E\} \to \textit{maximal totally isotropic subspace } S(x) \textit{ of } (S, \{\,|\,\})$$

[62] Cf., for example, C. Chevalley, *The Algebraic Theory of Spinors*, op. cit., or R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit.

in injective and realizes a natural embedding of $\tilde{Q}(E)$, the projective quadric associated with E, into the Grassmannian $G(S, \frac{1}{2}\dim S)$ of subspaces of S of dimension = $\frac{1}{2}\dim S$.

Now we are going to specify such an embedding in the case that $E = E_{r,s}$ is a pseudo-Euclidean space of type (r, s) with $m = \dim E = r+s = 2k+1$, $r-s \equiv \pm 3 \pmod 8$.

1.5.5.2 Case That *k* Is Even

We know that there exists on S a pseudoquaternionic neutral scalar product of type (p, p) with $p = \frac{1}{2}\dim_{\mathbf{H}} S = 2^{k-2} = 2^{\frac{m-1}{2}-2}$, associated with the involution τ and $C^+(E, q) \simeq \mathcal{L}_{\mathbf{H}}(S)$. As is known,[63] the Grassmannian of maximal totally isotropic subspaces of dimension k, denoted here by $G_k(S)$, of S a right vector space of dimension $n = p + q$, over $\mathbf{H}$, provided with a pseudoquaternionic scalar product of type (p, q) is homeomorphic to $(SpU(p) \times SpU(q))/(SpU(k) \times SpU(p-k) \times SpU(q-k))$. Then we obtain here that $G(S, \frac{1}{2}\dim S)$ is homeomorphic to $SpU(\frac{1}{2}\dim_{\mathbf{H}} S)$, whence we deduce the following result:

1.5.5.2.1 Theorem *For $m \geq 7$, $m = r + s = 2k + 1$, k even, $r - s \equiv \pm 3 \pmod 8$, the projective quadric $\tilde{Q}(E_{r,s})$ is naturally embedded into $SpU(2^{\frac{m-1}{2}-2})$.*

The set of subspaces of S that are positive maximal and then of dimension $\frac{1}{2}\dim_{\mathbf{H}} S = 2^{k-2}$ is an open set of the Grassmannian $G(S, \frac{1}{2}\dim S)$, which[64] *we agree to call the semi-Grassmannian of $(S, \{\,|\,\})$ and which we denote by $G_+(S)$. $G_+(S)$ is the classical simply connected symmetric space of type CII in Elie Cartan's list,*[65]

$$SpU(2^{k-2}, 2^{k-2})/SpU(2^{k-2}) \times SpU(2^{k-2}),$$

and $\tilde{Q}(E_{r,s})$ is embedded into the "boundary" of $G_+(S)$ in $G(S, 2^{k-2})$.

$G_+(S)$ can be identified with the symmetric space of involutions of $\mathcal{L}_{\mathbf{H}}(S) \simeq C^+(r, s)$ that commute with $\tau : \alpha\tau = \tau\alpha$ and that are strictly positive, which by definition[66] means that the real corresponding quadratic form on $C^+(r, s)$, $\mathrm{Tr}(l(x^\alpha x))$, is positive definite. As a matter of fact, if α is an involution, there exists (Theorem 1.3.3.3) an element $u \in \mathcal{L}_{\mathbf{H}}(S)$ determined up to a nonzero scalar factor such that

[63] Cf., for example. I. R. Porteus, op. cit., p. 237, Theorem 12-19 and p. 350, Proposition 17–46.

[64] R. Deheuvels, Rend. Sem. Mat. Univers. Politecn. Torino, op. cit.

[65] Cf., for example, S. Helgason, *Differential Geometry and Symmetric Spaces*, op. cit., or J. A. Wolf, *Spaces of Constant Curvature*, op. cit.

[66] Cf. A. Weil, op. cit.; I. Satake, op. cit.; R. Deheuvels, op. cit.

1. $a^\alpha = u^{-1} a^\tau u$, for any a in $\mathcal{L}_{\mathbf{H}}(S)$,
2. $u^\tau = u^\alpha = u$.

Now, $\alpha\tau = \tau\alpha$ implies that $u^2 = \lambda.1$, with λ a real number since $(u^2)^\tau = \lambda.1 = u^2 = \lambda.1$. Since $u^2 = u^\alpha u$ and since α is a positive involution, λ is a positive real number. Replacing u by $\lambda^{-1/2}u$, we obtain an involutive element $u \in \mathcal{L}_{\mathbf{H}}(S)$ such that $u^\tau = u^\alpha = u$ (with $u^2 = 1$), associated with the involution α of $\mathcal{L}_{\mathbf{H}}(S)$.

An associated scalar product $[\,|\,]$ on S is given by $[x|y] = \{ux|y\}$, corresponding to the involution α. The eigenspace of u for the eigenvalue 1 is then a maximal strictly positive subspace P_α of $(S, \{\,|\,\})$, since for any nonzero element x in P_α we have

$$\{x|x\} = \{ux|x\} = [x|x] > 0.$$

Then P_α is an element of $G_+(S)$. Conversely, the datum of such a subspace P determines its orthogonal Q, which is maximal and strictly negative. P and Q are, in fact, the eigenspaces relative to the eigenvalues 1 and -1 of an involutive element u of $\mathcal{L}_{\mathbf{H}}(S)$, which determines a strictly positive involution α by putting for any x in S, $x^\alpha = x^{\tau j(u)}$ with $\alpha\tau = \tau\alpha$. We recall that as usual, we put for any x, $x^{j(u)} = u^{-1}xu$ such that $j(u)j(v) = j(uv)$.

1.5.5.3 Case There *k* Is Odd

If k is odd, we know (1.5.1.2.3) that there exists on S a nondegenerate **H**-skew sesquilinear form b on S of maximal index $2^{\frac{m-1}{2}-2}$, and the corresponding special unitary group $SU_{2^{\frac{m-1}{2}-1}} = SO^*(2^{\frac{m-1}{2}})$.[67] It is known[68] that the Grassmannian of maximal totally isotropic subspaces of dimension $2^{\frac{m-1}{2}-2}$ of the complex vector space S provided with the skew-hermitian form h is homeomorphic to $U(2^{\frac{m-1}{2}-1})$ and according to results recalled by Porteous[69] the Grassmannian of maximal totally isotropic subspaces of dimension $2^{\frac{m-1}{2}-2}$ of the complex vector space S provided with the symmetric complex bilinear form a is homeomorphic to $O(2^{\frac{m-1}{2}})/U(2^{\frac{m-1}{2}-1})$. Thus we can deduce that the Grassmannian of maximal totally isotropic subspaces of dimension $2^{\frac{m-1}{2}-2}$ of the quaternionic right vector space S provided with the form b is homeomorphic to $O(2^{\frac{m-1}{2}})$. Then we have the following result:

1.5.5.3.1 Theorem *For $m \geq 7$, $m = r + s = 2k + 1$, k odd, $r - s \equiv \pm 3$ (mod 8), the projective quadric $\tilde{Q}(E_{r,s})$ is naturally embedded into $O(2^{\frac{m-1}{2}})$.*

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67 The unitary group of b is $U_{2^{\frac{m-1}{2}}}(C, h) \cap O(2^{\frac{m-1}{2}}, \mathbf{C})$, $n = \dim_{\mathbf{H}} S = 2^{\frac{m-1}{2}-1}$.

68 I. R. Porteus, op. cit., Theorem 12-12 p. 233, and Proposition 17-46, p. 350.

69 Ibid., Theorem 12-19, p. 237.

1.6 Real Structures on the Space S of Spinors for $C^+_{r,s}$, $m = 2k + 1$, $r - s \equiv \pm 1 \pmod 8$. Embedding of Corresponding Spin Groups and Associated Real Projective Quadrics

1.6.1 Involutions of the Real Algebra $\mathcal{L}_{\mathbf{R}}(S)$, where S is a Real Space over R of Even Dimension

1.6.1.1 Introductory Notes

Let S be a real vector space of even dimension n. Let b be a pseudo-Euclidean or symplectic scalar product on S. We know, (1.1.5.2), that we can define the adjunction $*$ in $\mathcal{L}_{\mathbf{R}}(S)$—relative to b—such that for any a, b in $\mathcal{L}_{\mathbf{R}}(S)$, for any λ in $\mathbf{R}$, we have

$$(a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*,\ (\lambda I)^* = \lambda I,\ (a^*)^* = a.$$

Now we want to show that any involution α of the real algebra $\mathcal{L}_{\mathbf{R}}(S)$ can be considered as the adjunction relative to a nondegenerate real symmetric or skew bilinear form on S.

Let $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_n\}$ be an arbitrary basis of S. Such a basis determines a pseudo-Euclidean scalar product on S, according to which ε is an orthonormal basis. Any element a in $\mathcal{L}_{\mathbf{R}}(S) = \mathcal{A}$ is represented by its matrix A in ε, and the adjunction $*$ is such that $A^* = {}^t A$.

If α is an involution of $\mathcal{A}$, according to 1.3.3.3, α can be written as $A^\alpha = U^{-1}({}^t A)U$ with ${}^tU = U$ or ${}^tU = -U$. If ${}^tU = U$, then ${}^tU = U^\alpha = U$ and if ${}^tU = -U$, then ${}^tU = U^\alpha = -U$. U is determined up to a nonzero scalar factor and represents in the basis ε the matrix of a nondegenerate bilinear form, symmetric (respectively skew) if ${}^tU = U$ (respectively ${}^tU = -U$), the adjunction of which is precisely α.

It is well known that a real symplectic vector space, the scalar product of which is denoted by $[\,|\,]$, is provided with a pseudo-Euclidean structure, the scalar product of which is denoted by $(\,|\,)$, if and only if there exists a transfer symplectic operator T, with $T^2 = -I$. S is then provided with a pseudo-hermitian structure, the scalar product of which is denoted by $\langle\,|\,\rangle$ and for any $x, y \in S$ we have $\langle x \mid y\rangle = (x \mid y) - i[x \mid y]$ with $(x \mid y) = -[Tx \mid y]$, or, equivalently, $[x \mid y] = (Tx \mid y)$. T is an orthogonal and symplectic operator.[70] Therefore we find that, as usual,

$$U(p,q) = SO(2p, 2q) \cap Sp(2(p+q), \mathbf{R}).$$

1.6.1.2 Properties of the Trace

Then the problem that appears is the following one: Since in 1.5.3, the form $(x, y) \to \mathrm{Tr}(l(x^\alpha y)$ is a nondegenerate symmetric bilinear form associated with the involution α of $\mathcal{A}$. Mutatis, mutandis, the demonstration is the same. Let ε' be

[70] Cf., for example, R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., pp. 429–430.

the basis of $\mathcal{L}_{\mathbf{R}}(S)$ associated with the basis ε of S, $\varepsilon' = \{\varepsilon_{ij}, 1 \leq i, j \leq n\}$, where ε_{ij} is the linear operator on S, the matrix of which in ε has all elements equal to zero except for that row i and column j, which is equal to 1.

If we choose an orthogonal basis ε for the pseudo-Euclidean scalar product on S, ε' is an orthogonal basis for the bilinear form $\mathrm{Tr}(l(x^\alpha y)$ on $\mathcal{A}$. If $H = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the matrix of the pseudo-Euclidean scalar product in ε with $\lambda_1, \ldots, \lambda_p > 0$ and $\lambda_{p+1}, \ldots, \lambda_{p+q} < 0$ $(p + q = n)$, we find easily that $X^\alpha = H^{-1}({}^t X)H$, and if $X = [x_{ij}]$, we find that

$$\mathrm{Tr}(l(X^\alpha X) = \sum \frac{\lambda_i}{\lambda_j} |x_{ij}|^2,$$

and that the signature of the quadratic form $X \to \mathrm{Tr}(l(X^\alpha X)$ on $\mathcal{A}$ is $(p^2 + q^2, 2pq)$.

1.6.2 Real Symplectic or Pseudo-Euclidean Structures on the Space S of Spinors for $C^+_{r,s}$, $m = r + s = 2k + 1$, $r - s \equiv \pm 1 \pmod 8$

Now, according to the fundamental table (1.4.2), $C^+(r, s) \simeq \mathcal{L}_{\mathbf{R}}(S)$, where $\dim_{\mathbf{R}} S = 2^{\frac{m-1}{2}}$. Following the method used in 1.5 and according to the results above (1.6.1), $A^\tau = U^{-1}({}^t A)U$ with ${}^t U = U^\tau = U$ or ${}^t U = U^\tau = -U$, where U is determined up to a nonzero scalar. As in 1.5, U is proportional to $\mathcal{J}_1$, the product of the elements of a basis of $E_1 = u^\perp$, where u is a vector of $E_{r,s}$ such that $(u \mid u) = \pm 1$ and E_1 is a pseudo-Euclidean space $E_{r,s-1}$ if $(u \mid u) = -1$ or $E_{s,r-1}$, if $(u \mid u) = 1$; $C^+(r, s)$ is then represented as the Clifford algebra $C(u^\perp)$. We obtain the following table:

Realization of $C^+_{r,s}$:

$C(u^\perp)$, $(u \mid u) = -1$	$C(u^\perp)$, $(u \mid u) = 1$
$E_1 = u^\perp$ of type $(r, s - 1)$	$E_1 = u^\perp$ of type $(s, r - 1)$
$\mathcal{J}_1$ product of the elements of the chosen basis of E_1	$\mathcal{J}'_1$ product of the elements of the chosen basis of E_1
$\mathcal{J}_1^2 = (-1)^{k+s-1}$	$\mathcal{J}'^2_1 = (-1)^{k+s}$
$m = r + s = 2k + 1$, $r - s = 1 + 8l$, $k \equiv s \pmod 4$	
$\mathcal{J}_1^2 = -1$	$\mathcal{J}'^2_1 = -1$
s even	
$\mathcal{J}_1^\tau = \mathcal{J}_1$, $N(\mathcal{J}_1) = -1$	$\mathcal{J}'^\tau_1 = \mathcal{J}'_1$, $N(\mathcal{J}'_1) = 1$
$\mathcal{J}_1 \notin \mathrm{Spin}(E_{r,s})$	$\mathcal{J}'_1 \in \mathrm{Spin}(E_{r,s})$
s odd	
$\mathcal{J}_1^\tau = -\mathcal{J}_1$, $N(\mathcal{J}_1) = 1$	$\mathcal{J}'^\tau_1 = -\mathcal{J}'_1$, $N(\mathcal{J}'_1) = -1$
$\mathcal{J}_1 \in \mathrm{Spin}(E, r)$	$\mathcal{J}'_1 \notin \mathrm{Spin}(E, r)$
$m = r + s = 2k + 1$, $r - s = -1 + 8l$, $k \equiv s - 1 \pmod 4$	
$\mathcal{J}_1^2 = -1$	$\mathcal{J}'^2_1 = -1$

s even

$\mathcal{J}_1^\tau = -\mathcal{J}_1,\ N(\mathcal{J}_1) = -1$ $\quad$ $\mathcal{J}_1'^\tau = -\mathcal{J}_1',\ N(\mathcal{J}_1') = 1$

$\mathcal{J}_1 \notin \mathrm{Spin}(E_{r,s})$ $\quad$ $\mathcal{J}_1' \in \mathrm{Spin}(E_{r,s})$

s odd

$\mathcal{J}_1^\tau = \mathcal{J}_1,\ N(\mathcal{J}_1) = 1$ $\quad$ $\mathcal{J}_1'^\tau = \mathcal{J}_1',\ N(\mathcal{J}_1') = -1$

$\mathcal{J}_1 \in \mathrm{Spin}(E_{r,s})$ $\quad$ $\mathcal{J}_1' \notin \mathrm{Spin}(E_{r,s})$

We deduce the following result:

1.6.2.1 Theorem *The space S of spinors associated with the Clifford algebras $C^+_{r,s}$, with $m = r + s = 2k + 1$, $r - s \equiv \pm 1$ (mod 8), possesses a natural real pseudo-Euclidean or symplectic structure according as k is even or odd, the scalar product of which is invariant by the spin group* Spin $E_{r,s}$. *For $m \geq 7$ we have the embedding of* Spin $E_{r,s}$ *into $SO(p, p)$ or $Sp(2p, \mathbf{R})$ with $p = 2^{\frac{m-1}{2}-1} = 2^{k-1}$ according as k is even or odd.*

The dichotomy k even or k odd results from the fact that $\mathcal{J}_1^\tau = \mathcal{J}_1$ (or $\mathcal{J}_1'^\tau = \mathcal{J}_1'$), respectively $\mathcal{J}_1^\tau = -\mathcal{J}_1$ (or $\mathcal{J}_1'^\tau = -\mathcal{J}_1'$), according as k is even or odd, which determines the choice of the pseudo-Euclidean, respectively symplectic, scalar product on S.

1.6.2.1 The Case of k Even

If k is even, the same technique as in 1.5 shows that the quadratic form defined on $C^+(r, s) = \mathcal{L}_{\mathbf{R}}(S)$ is neutral and that S is a pseudo-Euclidean neutral vector space of type $(2^{\frac{m-1}{2}-1}, 2^{\frac{m-1}{2}-1})$. The unitary group of automorphisms of S that leave this scalar product invariant consists of elements $u \in C^+(r, s)$ such that $u^\tau u = 1$ and thus contains Spin $E_{r,s}$. Moreover, we want to show that Spin $E_{r,s}$ is embedded into $SO(2^{\frac{m-1}{2}-1}, 2^{\frac{m-1}{2}-1})$, for $m \geq 7$, i.e., that the elements of Spin $E_{r,s}$ have a determinant equal to 1, as linear operators of S.

If $r - s = 1 + 8l$, s is even, and u is such that $(u \mid u) = -1$, we can take again the above demonstration of 1.5 for the case $r - s \equiv \pm 3$ (mod 8), which shows that any element g in Spin $E_{r,s}$ is the product of elements that have determinant equal to ± 1 in S. Spin $E_{r,s}$ is then embedded into the subgroup of $O(2^{k-1}, 2^{k-1})$ consisting of elements with determinant equal to ± 1 in S, but according to the connectedness of Spin $E_{r,s}$ all these elements necessarily have determinant equal to 1. If $(u \mid u) = 1$ we again use the same method.

If $r - s = -1 + 8l$, s is odd, and if u is such that $(u|u) = -1$, we can also take the following route. Since m is odd, $\mathcal{J} = e_1 \ldots e_{r+s}$ belongs to the center of $C_{r,s}$ and $\mathcal{J}^2 = (-1)^{k+s} = -1$. Let us take a vector u_1 such that $u_1^2 = -1$. Then $(\mathcal{J} u_1)^2 = 1$, $\mathcal{J} u_1$ is an involution of S, and then has determinant equal to ± 1.

If u_1 and v_1 belong to $V = E_{r,s}$, then $\mathcal{J} u_1 \mathcal{J} v_1 = u_1 \mathcal{J}^2 v_1 = -u_1 v_1$, whence we can deduce that any element g of Spin $E_{r,s}$ is a product of elements with determinant $= \pm 1$ in S and the connectedness of Spin V yields the result. If $(u \mid u) = 1$, we take the same route.

1.6.2.2 The Case of k Odd

(b) *If k is odd*, the symplectic structure comes naturally.

1.6.2.3 Remark According to the table above (1.6.2), we can always find a realization of $C^+(r, s)$ such that $\mathcal{J}_1$ or $\mathcal{J}_1'$ is in the symplectic case an element of Spin $E_{r,s}$, the square of which is equal to -1, and thus a symplectic automorphism of S with square equal to -1. If $r - s = 1 + 8l$ and s is odd, this element is $\mathcal{J}_1$. If $r - s = -1 + 8l$ and s is even, this element is $\mathcal{J}_1'$. $\mathcal{J}_1$ (respectively $\mathcal{J}_1'$) is a transfer operator that provides S with a pseudo-Euclidean neutral structure associated with the symplectic structure (the neutrality of the structure can be shown as in 1.5.3).

Furthermore, $\mathcal{J}_1$ belongs to the anticenter of a Clifford algebra $C_{r,s-1}$ of a vector space of precise dimension $2k = r + s - 1$ and $\mathcal{J}_1'$ belongs to the anticenter of a Clifford algebra $C_{s,r-1}$ of a vector space of dimension $2k$. Thus, $\mathcal{J}_1$ and $\mathcal{J}_1'$ commute with any element in Spin V. Then, the pseudo-Euclidean scalar product ε classically defined for any x, y in S by $\varepsilon(x, y) = -[\mathcal{J}_1 x \mid y]$, respectively $\varepsilon(x, y) = -[\mathcal{J}_1' x \mid y]$, is such that for any g in Spin V, for any x, y in S, $\varepsilon(gx, gy) = -[\mathcal{J}_1 gx \mid gy] = -[g\mathcal{J}_1 x \mid gy] = -[\mathcal{J}_1 x \mid y] = \varepsilon(x, y)$, since g is in Spin V and leaves $[\,|\,]$ invariant (and the same is true with $\mathcal{J}_1'$). Thus, if k is odd and $r - s = 1 + 8l$ (s odd) and if k is odd and $r - s = -1 + 8l$ (s even), Spin V is embedded into $SU(2^{k-2}, 2^{k-2})$, for $m \geq 7$, which is a refinement of the result above.

1.6.3 Embedding of Corresponding Projective Quadrics

We obtain the following result:

1.6.3.1 Theorem *For $m \geq 7, m = r+s = 2k+1, r-s \equiv \pm 1$ (mod 8), the projective quadric $\tilde{Q}(E_{r,s})$ is embedded into the group $O(2^{k-1})$ if k is even and into the group $U(2^{k-1})/O(2^{k-1})$ and even into $U(2^{k-1})$ if k is odd.*

The embedding can be made as in 1.5.5. If k is even we know[71] that the Grassmannian $G_{k'}(S)$ of a totally isotropic subspaces of dimension k' of the pseudo-Euclidean space S of type (p, q) is homeomorphic to $(O(p) \times O(p))/(O(k') \times O(p - k') \times O(q - k'))$, which gives a homeomorphism of $G(S, \frac{1}{2}\dim S)$ with $O(2^{k-1})$ (with notation of 1.5.5). If k is odd, we know[72] that the Grassmannian $G(S, \frac{1}{2}\dim S)$ is homeomorphic to $U(2^{k-1})/O(2^{k-1})$ and even in $U(2^{k-1})$ according to the remark of 1.6.2.3 above. If k is even, the same approach as in 1.5.5 shows that the set of subspaces of S that are maximal and strictly positive, and thus of dimension 2^{k-1}, is an open set of the Grassmannian $G(S, 2^{k-1})$ called, by definition, the semi-Grassmannian of $(S, (\,|\,)$ and denoted by $G_+(S)$. $G_+(S)$ is the classical symmetric space $SO^+(2^{k-1}, 2^{k-1})/SO(2^{k-1}) \times SO(2^{k-1})$ of type BDI in Elie Cartan's list.[73]

[71] For example, I. R. Porteous, op. cit., p. 237 and p. 350.

[72] Ibid., p. 233.

[73] S. Helgason, op. cit., p. 394, for example.

As in 1.5.5, $G_+(S)$ can be identified with the symmetric space of involutions α of $C^+(r, s)$ that commute with τ and that are strictly positive. If k is odd, according to the existence of the pseudo-Euclidean structure associated with the symplectic structure, we obtain analogous conclusions with respect to a previous remark (1.6.2.3).

1.7 Study of the Cases $r - s \equiv 0 \pmod 8$ and $r - s \equiv 4 \pmod 8$

According to the fundamental table (1.4.2), if $r-s \equiv 0 \pmod 8$ $C^+(r, s)$ is a semisimple algebra, isomorphic to the direct sum of two algebras isomorphic to $m(2^{\frac{[m-1]}{2}}, \mathbf{R})$, and if $r-s \equiv 4 \pmod 8$, $C^+(r, s)$ is isomorphic to the direct sum of two algebras isomorphic to $m(2^{\frac{[m-1]}{2}-1}, \mathbf{H})$. Now,[74] if m is even and $m \equiv 2 \pmod 4$, τ interchanges the two simple components of $C^+(r, s)$, and if $m \equiv 0 \pmod 4$, τ leaves invariant each of the two components of $C^+(r, s)$ and induces on each of them an involution of the first type,[75] i.e., fixes all elements in the center of the algebra. Let us put for the algebra $\mathcal{A}$, $P_\tau(\mathcal{A}) = \{x, x \in \mathcal{A} \mid x^\tau = x\}$, $\mathcal{J}_\tau(\mathcal{A}) = \{x, x \in \mathcal{A} \mid x^\tau = -x\}$, spaces of even elements, respectively odd elements, for the involution τ of the algebra $\mathcal{A}$. We recall that for any element $e_I = e_{i_1} \cdots e_{i_p}$ $(1 \leq i_1 < i_2 < \cdots < i_p \leq m)$ of the basis of $C_{r,s}$ associated with the basis $\{e_i\}_{1 \leq i \leq m}$ of $E_{r,s}$ we have $e_A^\tau = (-1)^{p(p-1)/2} e_A$. One easily verify that

$$\begin{array}{ll} P_\tau(C_{r,s}) = \oplus C_p(E_{r,s}) & \mathcal{J}_\tau(C_{r,s}) = \oplus C_p(E_{r,s}) \\ \quad p \equiv 0 \text{ or } 1 \pmod 4 & \quad p \equiv 2 \text{ or } 3 \pmod 4 \\ P_\tau(C^+_{r,s}) = \oplus C_p(E_{r,s}) & \mathcal{J}_\tau(C^+_{r,s}) = \oplus C_p(E_{r,s}) \\ \quad p \equiv 0 \pmod 4 & \quad p \equiv 2 \pmod 4 \end{array}$$

where as usual, $C_p(E_{r,s})$ denotes the subspace of p-vectors of $E_{r,s}$.[76] According to Weil,[77] if $(x, y) \to \mathrm{Tr}(l(x^\tau y))$ is the nondegenerate symmetric bilinear form defined on $C^+_{r,s}$, $C^+_{r,s}$ is the direct orthogonal sum of $P_\tau(C^+_{r,s})$ and $\mathcal{J}_\tau(C^+_{r,s})$: $C^+_{r,s} = P_\tau(C^+_{r,s}) \oplus \mathcal{J}_\tau(C^+_{r,s})$. We are going to use classical results concerning the structure of Clifford algebras.[78]

1.7.1 Study of the Case $r - s \equiv 0 \pmod 8$

The even Clifford algebra $C^+_{r,s}$ for a standard pseudo-Euclidean space $E_{r,s}$, $m = r + s = 2k$, $r - s \equiv 0 \pmod 8$) is the direct sum of two simple algebras C_1 and C_2

[74] I. Satake, op. cit., p. 281.

[75] Ibid., p. 268.

[76] Cf., for example, R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., p. 327.

[77] A. Weil, op. cit. lemma 1, p. 603.

[78] Cf. R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, chapter VII ("theorem VIII-8" p. 317 and "proposition VII-18" p. 334, chapters VIII-13 and VIII-14).

isomorphic to its even subalgebra $(C^+_{r,s})^+$, isomorphic itself to $m(2^{k-1}, \mathbf{R})$ according to the fundamental table (1.4.2).[79]

More precisely, $C^+_{r,s} = C_1 \oplus C_2$ with $C_1 = (C^+_{r,s})^+\varepsilon_1$, $C_2 = (C^+_{r,s})^+\varepsilon_2$, where $\varepsilon_1 = \frac{1}{2}(1+\mathcal{J}_1)$, $\varepsilon_2 = \frac{1}{2}(1-\mathcal{J}_1)$, if we denote by $\mathcal{J}_1$ the product of the elements of the basis of $u^\perp = E_1$, chosen for the realization of $C^+_{r,s}$,[80] with $\varepsilon_1 + \varepsilon_2 = 1$, $\varepsilon_1\varepsilon_2 = 0$. C_1 and C_2 are both two-sided ideals of $C^+_{r,s}$. Now let us consider S, the space of spinors for $C^+_{r,s}$, the minimal faithful module, defined up to an isomorphism, of the algebra $C^+_{r,s}$. According to Deheuvels,[81] since $C^+_{r,s}$ is semisimple, S is a direct sum of two nonisomorphic submodules, spaces of spinors called semispinors, $S = S_1 \oplus S_2$. We obtain the following table:

$$m = r + s = 2k, \; r - s = 8l, \; k \equiv s \pmod 4);$$

Realization of $C^+_{r,s}$:

$C(E_1) = C(u^\perp)$	$C(E_1) = C(u^\perp)$
$(u \mid u) = -1$, $C(E_1) = C(E_{r,s-1})$	$(u \mid u) = 1$, $C(E_1) = C(E_{s,r-1})$
$\mathcal{J}_1$ product of the elements of the chosen basis of $E_1 = u^\perp$	$\mathcal{J}'_1$ product of the elements of the chosen basis of $E_1 = u^\perp$
$\mathcal{J}_1^2 = (-1)^{k+s}$, $\mathcal{J}_1^\tau = (-1)^{k-1}\mathcal{J}_1$	$\mathcal{J}_1'^2 = (-1)^{k+s}$, $\mathcal{J}_1'^\tau = (-1)^{k-1}\mathcal{J}'_1$

1.7.1.1 k even (r and s even), $m \equiv 0 \pmod 4$

$\mathcal{J}_1^2 = 1$, $\mathcal{J}_1^\tau = -\mathcal{J}_1$, $N(\mathcal{J}_1) = -1$	$\mathcal{J}_1'^2 = 1$, $\mathcal{J}_1'^\tau = -\mathcal{J}'_1$, $N(\mathcal{J}'_1) = -1$
$\tilde{\mathcal{J}}_1^\tau = -\tilde{\mathcal{J}}_1$	$\tilde{\mathcal{J}}_1'^\tau = -\tilde{\mathcal{J}}'_1$

Let us write the two-sided ideals C_1 and C_2 of $C^+_{r,s}$:

$$C_1 = (C^+_{r,s})^+\varepsilon_1, \text{ respectively } (C^+_{r,s})^+\varepsilon'_1,$$
$$C_2 = (C^+_{r,s})^+\varepsilon_2, \text{ respectively } (C^+_{r,s})^+\varepsilon'_2$$

with

$$\varepsilon_i \text{ or } \varepsilon'_i = \frac{1}{2}\left(1 \pm \begin{matrix}\mathcal{J}_1 \\ \mathcal{J}'_1\end{matrix}\right) \text{ for } i = 1, 2.$$

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[79] An elementary calculation shows that if we realize $C^+_{r,s}$ as $C_{r,s-1}$, $(C^+_{r,s})^+$ realized as $C_{r,s-2}$ or $C_{s-1,r-1}$, in both cases $(C^+_{r,s})^+$ is isomorphic to $m(2^{k-1}, \mathbf{R})$, and if we realize $C^+_{r,s}$ as $C_{s,r-1}$, $(C^+_{r,s})^+$ realized as $C_{s,r-2}$ or $C_{r-1,s-1}$, in both cases $(C^+_{r,s})^+$ is isomorphic to $m(2^{k-1}, \mathbf{R})$.

[80] Cf. R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., "théorème VIII-8," p. 317.

[81] Ibid., "chapitre VIII-14."

We know that τ leaves invariant C_1 and C_2. We are going to use classical results concerning faithful representations of semisimple algebras.[82] We put S_1 as a C_1-simple module, S_2 as C_2-simple module, and for $j \neq i$, $(i, j = 1, 2)$, $C_j S_i = 0$, and S_i is isomorphic as a C_i-simple module, with a simple left ideal of C_1: $\mathcal{H}_i = C_i \tilde{e}_i$, where $\tilde{e}_i$ is an idempotent of $\mathcal{H}_i$. C_1 is a central simple algebra, the C_1-simple module of which is S_1, and C_1 can be identified with the real algebra $\mathcal{L}_{\mathbf{R}}(S) \simeq m(2^{k-1}, \mathbf{R})$, to which the results of 1.6 can be applied.

Let τ be chosen for involution of C_1. Let $\varepsilon^1 = \{\varepsilon_i^1\}_{1 \leq i \leq n}$ ($n = \dim_{\mathbf{R}} S_1$) be an arbitrary basis of S_1. This basis determines on S_1 a standard Euclidean scalar product, according to which ε^1 is an orthogonal basis. Any element of $C_1 = \mathcal{L}_{\mathbf{R}}(S_1)$ is represented by its matrix in ε^1, and the corresponding adjunction $*$ is such that $A^* = {}^tA$. As in 1.6, $A^\tau = U^{-1}({}^tA)U$ with ${}^tU = U$ or ${}^tU = -U$. If ${}^tU = U$, then ${}^tU = U^\tau = U$, and if ${}^tU = -U$, then ${}^tU = U^\tau = -U$. The element U that conducts the inner automorphism is determined up to a nonzero scalar factor, as above (1.6), and is proportional to $\tilde{\mathcal{J}}_1$ (or $\tilde{\mathcal{J}}_1'$), the product of the elements of the basis chosen for the realization of $(C_{r,s}^+)^+$ isomorphic to C_1. Now we have $\tilde{\mathcal{J}}_1^\tau = -\tilde{\mathcal{J}}_1$, respectively $\tilde{\mathcal{J}}_1'^\tau = -\tilde{\mathcal{J}}_1'$, and then U determines on S_1 a symplectic scalar product the adjunction of which is τ. Moreover, the nondegenerate symmetric bilinear form $(x, y) \to Tr(l(x^\tau y))$ defined on C_1 is a neutral form. The demonstration is the same as in 1.6 and can be also made for C_2 and S_2.

Furthermore, $S = S_1 \oplus S_2$ is provided with a symplectic scalar product defined for any $z = z_1 + z_2$ (with $z_i \in S_i$ for $i = 1, 2$), $z' = z_1' + z_2'$ (with $z_i' \in S_i$ for $i = 1, 2$), by $[z \mid z'] = [z_1 \mid z_1'] + [z_2 \mid z_2']$. Since any element $g \in \mathrm{Spin}(E_{r,s})$ can be written $g = g_1 + g_2$, $g_1 \in C_1$, $g_2 \in C_2$, and since $g_2 \cdot S_1 = 0$ and $g_1.S_2 = 0$, and since the group of automorphisms of S_1 (respectively S_2) leaving invariant the symplectic scalar product consists of elements u in $C_{r,s}^+$ such that $u^\tau u = 1$, we can deduce that $\mathrm{Spin}(E_{r,s})$ is included in $Sp(2p, \mathbf{R})$ with $p = 2^{k-1}$. As in 1.6, we have the following embedding of $\tilde{Q}(E_{r,s})$, the corresponding projective quadric, into $U(2^{k-1})/O(2^{k-1})$. Thus, we have the following results:

1.7.1.1.1 Theorem *For $m \geq 8$, $r - s \equiv 0$ (mod 8), $m = r + s = 2k$, k even (r and s even) and then $m \equiv 0$ (mod 4). $\mathrm{Spin}(E_{r,s})$ is embedded into $Sp(2p, \mathbf{R})$ with $p = 2^{k-1}$; $\tilde{Q}(E_{r,s})$ is embedded into $U(2^{k-1})/O(2^{k-1})$.*

1.7.1.1.2 Fundamental Remark

Since k is even, r and s even, we can always find a realization of $(C_{r,s}^+)^+$: $C_{r,s-2}$, or respectively $C_{s,r-2}$, such that for $\tilde{\mathcal{J}}_1$, respectively for $\tilde{\mathcal{J}}_1'$, we have $\tilde{\mathcal{J}}_1^2 = -1$, respectively $\tilde{\mathcal{J}}_1'^2 = -1$. Moreover, $\tilde{\mathcal{J}}_1$ and $\tilde{\mathcal{J}}_1'$, since the dimension is even, both belong to the anticenter of $(C_{r,s}^+)^+$ and then commute with even elements in $(C_{r,s}^+)^+$. Since we have the above decomposition $g = g_1 + g_2$ ($g_i \in C_i$, for $i = 1, 2$) for any g in $\mathrm{Spin}(E_{r,s})$,

[82] Cf., for example, R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., "proposition VIII-13-C" p. 341, "théorème VIII-13-E" p. 343, "théorème VIII-4-A" pp. 345–346.

we can deduce on S_1 as on S_2 the existence of a "structure of type $SO(p_1, p_1)$," and consequently, Spin V is embedded into $SU(2^{k-1}, 2^{k-1})$, following the same route as in 1.6 and using a consideration of determinant equal to 1. Therefore, as in 1.6, the corresponding projective quadric $\tilde{Q}(E_{r,s})$ is embedded into $U(2^{k-1})$ for $k \geq 4$.

1.7.1.2 k odd (r and s odd) $m = 2k \equiv 2 \pmod 4$

We obtain with the above notation

$$\mathcal{J}_1^\tau = \mathcal{J}_1, N(\mathcal{J}_1) = \mathcal{J}_1^2 = 1, \qquad \mathcal{J}_1'^\tau = \mathcal{J}_1', N(\mathcal{J}_1') = 1 = \mathcal{J}_1'^2,$$
$$\tilde{\mathcal{J}}_1^\tau = \tilde{\mathcal{J}}_1, \qquad \tilde{\mathcal{J}}_1'^\tau = \tilde{\mathcal{J}}_1'.$$

We know that τ interchanges C_1 and C_2, the two simple components of $C_{r,s}^+$, since $m \equiv 2 \pmod 4$.[83]

Moreover, C_1 and C_2 are orthogonal for the real symmetric bilinear form: $(x, y) \to \mathrm{Tr}(l(x^\tau y))$. In the first case, case of $\mathcal{J}_1$, we now have $(1 + \mathcal{J}_1)^\tau = 1 + \mathcal{J}_1$, $(1 - \mathcal{J}_1)^\tau = 1 - \mathcal{J}_1$, and since $(1 + \mathcal{J}_1)^\tau (1 - \mathcal{J}_1)^\tau = (1 + \mathcal{J}_1)(1 - \mathcal{J}_1) = 0$ since $\mathcal{J}_1^2 = 1$, if $x = \frac{1}{2}(1 + \mathcal{J}_1)a \in C_1$, $y = \frac{1}{2}(1 - \mathcal{J}_1)b \in C_2$, we find that $x^\tau y = 0$. The same can be done in the case $\mathcal{J}_1'$.

According to classical results concerning faithful representations of semisimple algebras[84] already given, S_1 is a C_1-simple module, S_2 is a C_2-simple module, and if $j \neq i$ $(i, j = 1, 2)$, $C_j.S_i = 0$ and S_i is isomorphic to a left simple ideal $\mathcal{H}_i = C_i \tilde{e}_i$ of C_i—where $\tilde{e}_i$ is an idempotent of $\mathcal{H}_i$, for $i = 1, 2$—according to its structure as a C_i-module.

As in the above case 1.7.1.1, C_1 is a central simple algebra, the C_i-simple module of which is S_1, and C_1 can be identified with $\mathcal{L}_{\mathbf{R}}(S_1) = m(2^{k-1}, \mathbf{R})$. With previous notation, any element of C_1 is represented by its matrix in ε^1, an arbitrary basis of S_1, and $A^\tau = U^{-1}({}^t A)U$ now with $U^\tau = {}^t U = U$, and U is proportional to $\tilde{\mathcal{J}}_1$, respectively to $\tilde{\mathcal{J}}_1'$. S_1 is then provided with a pseudo-Euclidean scalar product. As in 1.6 above, if (p, q) denotes the signature of such a pseudo-Euclidean scalar product defined on S_1, we verify that the signature of the quadratic form $X \to \mathrm{Tr}(l(X^\tau X))$ defined on C_1 is $(p^2 + q^2, 2pq)$, and the same approach as in 1.6.5 shows that such a quadratic form is a neutral one. Therefore, the pseudo-Euclidean scalar product defined on S_1 is also a neutral one.

The same is true for C_2 and S_2. $S = S_1 \oplus S_2$ is then provided with a pseudo-Euclidean neutral scalar product defined for any $z = z_1 + z_2$, $z' = z_1' + z_2'$, where z_i and z_i' belong to S_i for $i = 1, 2$, by $(z \mid z') = (z_1 \mid z_1') + (z_2 \mid z_2')$. As in 1.7.1.1 above, we can write $g \in \mathrm{Spin}(E_{r,s})$ as $g = g_1 + g_2$ with $g_i \in C_i$ for $i = 1, 2$; and for $j \neq i$ we have $g_i S_j = 0$ $(i, j = 1, 2)$.

Since the group of automorphisms of S_1, respectively S_2, that leave invariant this pseudo-Euclidean neutral scalar product consists of elements u in $C_{r,s}^+$ such that

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[83] I. Satake, op. cit.

[84] Cf., for example, R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., "chapitres VIII-13, VIII-14."

$u^\tau u = 1$, using the same route as in 1.5 we show easily that any element in $\mathrm{Spin}(E_{r,s})$ is a product of elements with determinant equal to ± 1 in S_1, respectively S_2, and for a reason of connectedness, all these elements have necessarily determinant equal to 1. Thus, $\mathrm{Spin}(E_{r,s})$ is included into $SO(p_1, p_2)$ with $p_1 = 2^{k-1}$, and the corresponding projective quadric $\tilde{Q}(E_{r,s})$ is embedded into $O(2^{k-1})$ (for $m \geq 8$). We have then obtained the following result:

1.7.1.2.1 Theorem *For $m \geq 8$, $m = r+s = 2k$, k odd (r and s odd), $m \equiv 2 \pmod 4$,* $\mathrm{Spin}(E_{r,s})$ *is embedded into $SO(p_1, p_1)$ with $p_1 = 2^{k-1}$, and $\tilde{Q}(E_{r,s})$ is embedded into $O(2^{k-1})$.*

1.7.2 Study of the Case $r - s \equiv 4 \pmod 8$

According to the fundamental table (1.4.2), the Clifford algebra $C^+_{r,s}$ in the case $m = r + s = 2k$, $r - s \equiv 4 \pmod 8$ is the direct sum of two simple algebras, both isomorphic to its even subalgebra $(C^+_{r,s})^+$ and also isomorphic to $m(2^{k-2}, \mathbf{H})$. With the same notation as above, $C^+_{r,s} = C_1 \oplus C_2$, $C_1 = (C^+_{r,s})^+\varepsilon_1$, $C_2 = (C^+_{r,s})^+\varepsilon_2$. Let S be the space of spinors for $C^+_{r,s}$ a semisimple minimal faithful module over $C^+_{r,s}$, the direct sum of two simple nonisomorphic submodules S_1 and S_2. We have the following table:

$$\boldsymbol{m = r + s = 2k,\ r - s = 4 + 8l,\ k \equiv s + 2 \pmod 4}$$

Realization of $C^+_{r,s}$: $C(E_{r,s-1})$	$C^+_{r,s}$: $C(E_{s,r-1})$
$\boldsymbol{E_{r,s-1} = E_1 = u^\perp}$	$\boldsymbol{E_{s,r-1} = E_1 = u^\perp}$
$\mathcal{J}_1$ product of the elements of the chosen basis of $u^\perp$	$\mathcal{J}'_1$ product of the elements of the chosen basis of $u^\perp$
$\mathcal{J}_1^2 = (-1)^{k+s}$, $\mathcal{J}_1^\tau = (-1)^{k-1}\mathcal{J}_1$	$\mathcal{J}_1'^2 = (-1)^{k+s}$, $\mathcal{J}_1'^\tau = (-1)^{k-1}\mathcal{J}'_1$
$\tilde{\mathcal{J}}_1$ product of the elements of the chosen basis for the realization of $(C^+_{r,s})^+$	$\tilde{\mathcal{J}}'_1$ product of the elements of the chosen basis for the realization of $(C^+_{r,s})^+$
$\tilde{\mathcal{J}}_1^\tau = (-1)^{k+1}\mathcal{J}_1$	$\tilde{\mathcal{J}}_1'^\tau = (-1)^{k+1}\tilde{\mathcal{J}}'_1$

1.7.2.1 k Even, r and s Even, $m \equiv 0 \pmod 4$

The route is the same as in 1.7.1.1. C_1 is a central simple algebra the C_1-simple module of which is S_1, and C_1 can be identified with the real algebra $\mathcal{L}_{\mathbf{R}}(S_1) \simeq m(2^{k-2}), \mathbf{H})$ for which previous results (1.5) can be applied. C_1 is identified with the real algebra of linear operators of the quaternionic right vector space S_1. We choose τ for involution of C_1. Let $\varepsilon^1 = \{\varepsilon^1_i\}_{1 \leq i \leq n}$, $n = \dim_{\mathbf{H}}(S_1) = 2^{k-2}$, be an arbitrary basis of S_1. This basis determines on S_1 a quaternionic scalar product for which ε^1 is an orthogonal basis. Any element a in C_1 is represented by its matrix in ε^1, and the corresponding adjunction $*$ is such that with the same notation as above (1.5), $A^* =^t A^\nu$. As in 1.5,

$A^\tau = U^{-1}({}^t A^\nu)U$ with ${}^tU^\nu = U$ or ${}^tU^\nu = -U$. If ${}^tU^\nu = U$, then ${}^tU^\nu = U^\tau = U$, and if ${}^tU^\nu = -U$, then ${}^tU^\nu = U^\tau = -U$.

The element U that conducts the inner automorphism is proportional to $\tilde{\mathcal{J}}_1$, respectively $\tilde{\mathcal{J}}'_1$. Since now we have $\tilde{\mathcal{J}}_\tau = -\tilde{\mathcal{J}}_1$, respectively $\tilde{\mathcal{J}}'_\tau = -\tilde{\mathcal{J}}'_1$, U determines on S_1 a nondegenerate **H**-skew sesquilinear form on S_1, of maximal index $[\frac{n}{2}]$. Furthermore, the nondegenerate real symmetric corresponding bilinear form $(x, y) \to \mathrm{Tr}(l(x^\tau y))$ defined on C_1 is a neutral one, using the same approach as in 1.5. The same is true for C_2 and S_2, where we find that $S = S_1 \oplus S_2$ is provided with a nondegenerate **H**-skew sesquilinear form b defined by $b(z, z') = b_1(z_1, z'_1) + b_2(z_2, z'_2)$, where $z = z_1 + z_2$, $z' = z'_1 + z'_2$, z_i and z'_i in S_i, $i = 1, 2$, and b_i is the restriction to S_i of the above sesquilinear form. As in 1.7.1 we find that $\mathrm{Spin}(E_{r,s})$ is embedded into $U_{n_1}(S, b)$ and even in $SU_{n_1}(S, b) = SO^*(2n_1)$, with $n_1 = \dim_{\mathbf{H}} S = 2^{k-1}$, following the same approach as in 1.5, and that the corresponding projective quadric $\tilde{Q}(E_{r,s})$ is embedded into $O(2^k)$, for $k \geq 3$. Then we have obtained the following:

1.7.2.1.1 Theorem *For $m \geq 6$, $m = r + s = 2k$, k even, r and s even, $m \equiv 0$ (mod 4), $\mathrm{Spin}(E_{r,s})$ is embedded into $SO^*(2^k)$, $\tilde{Q}(E_{r,s})$ is embedded into $O(2^k)$.*

1.7.2.2 *k* Odd, *r* and *s* Odd, $m \equiv 2 \pmod 4$

Then we obtain

$$\mathcal{J}_1^2 = 1,\ \mathcal{J}_1^\tau = \mathcal{J}_1, \qquad \mathcal{J}_1'^2 = 1,\ \mathcal{J}_1'^\tau = \mathcal{J}_1^\tau,$$
$$\tilde{\mathcal{J}}_1^\tau = \tilde{\mathcal{J}}_1, \qquad \tilde{\mathcal{J}}_1'^\tau = \tilde{\mathcal{J}}'_1.$$

As in 1.7.1.2, C_1 and C_2 are interchanged by τ and are orthogonal for the real bilinear symmetric form $(x, y) \to \mathrm{Tr}(l(x^\tau y))$. S_1 is a C_1-simple module, S_2 is a C_2-simple module, and for $j \neq i$, $C_j.S_i = 0$ for $i, j = 1, 2$. C_1 is a central simple algebra the C_1-simple module of which is S_1, and C_1 can be identified with $\mathcal{L}_{\mathbf{R}}(S_1) = m(2^{k-2}, \mathbf{H})$.

Any element in C_1 is represented by its matrix in ε^1, a basis of S_1, and $A^\tau = U^{-1}({}^t A^\nu)U$ with ${}^tU^\nu = U = U^\tau$, and U is proportional to $\tilde{\mathcal{J}}_1$, respectively $\tilde{\mathcal{J}}'_1$. S_1 is then provided with a pseudoquaternionic scalar product of signature (p, q). As in 1.5, we verify that the signature of the corresponding real quadratic form $X \to \mathrm{Tr}(l(X^\tau X))$ is $(4(p^2 + q^2), 8pq)$. Moreover, this quadratic form is a neutral one, following the same route as in 1.5. The same can be done for S_2 and C_2. We have obtained the following result:

1.7.2.2.1 Theorem *For $m \geq 6$, $m = r + s = 2k$, k odd, r and s odd, $m \equiv 2$ (mod 4), $\mathrm{Spin}(E_{r,s})$ is embedded into $SpU(p, p) \subset SU(2p, 2p)$ with $p = 2^{k-2}$; $\tilde{Q}(E_{r,s})$ is embedded into $SpU(2^{k-2})$.*

1.8 Study of the Case $r - s \equiv \pm 2 \pmod 8$

In these cases, $C^+_{r,s} = \mathcal{A}$ is isomorphic to $m(2^{[\frac{m-1}{2}]}, \mathbf{C})$ according to the fundamental table (1.4.2) ($m = r + s = 2k$). $\mathcal{A}$ can be identified with the central simple complex algebra of linear operators of a complex vector space of dimension $2^{[\frac{m-1}{2}]}$.

1.8.1 Involutions on $\mathcal{A} = \mathcal{L}_{\mathbf{C}}(S)$, where S is a Complex Vector Space of Dimension n

Let $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_n\}$ be an arbitrary basis of S. Such a basis determines on S a hermitian standard scalar product for which ε is an orthonormal basis. Any element a in $\mathcal{A}$ is represented by its matrix A in ε and the corresponding adjunction $*$ is such that $A^* =^t \bar{A}$. Any involution α of $\mathcal{A}$ is such that according to the fundamental Theorem 1.3.3.3, $A^\alpha = U^{-1}({}^t\bar{A})U$ with ${}^t\bar{U} = U$ or ${}^t\bar{U} = -U$. If ${}^t\bar{U} = U$, U is the matrix in ε of a nondegenerate $\mathbf{C}$-sesquilinear form that determines a pseudo-hermitian scalar product the adjunction of which is precisely α. If ${}^t\bar{U} = -U$, U is the matrix in ε of a nondegenerate skew-hermitian sesquilinear form the adjunction of which is α. (This form is, in fact, of maximal index $[\frac{n}{2}]$.) It is classical[85] that then ib is a pseudo-hermitian sesquilinear form.

With the same notation as above, $\mathcal{J}^\tau = (-1)^k\mathcal{J}$, $\mathcal{J}_1^\tau = (-1)^{k-1}\mathcal{J}_1$, and $\mathcal{J}'^\tau = (-1)^{k-1}\mathcal{J}_1'$. The dichotomy k even and k odd appears naturally, since the element that conducts the inner automorphism is proportional to $\mathcal{J}_1$ (respectively $\mathcal{J}_1'$). If k is even, S is provided with a nondegenerate skew-hermitian form b and with ib a nondegenerate pseudo-hermitian form. If k is odd, S is provided with a pseudo-hermitian scalar product.

As above, we are going to study the following problem: If the involution α on the complex central simple algebra $\mathcal{A} = \mathcal{L}_{\mathbf{C}}(S)$ is associated with a pseudo-hermitian scalar product, determine the signature (p, q) of this pseudo-hermitian scalar product, the adjunction of which is α.

1.8.2 Associated Form with an Involution α of $\mathcal{A} = \mathcal{L}_{\mathbf{C}}(S)$

The same approach as above leads to the following result:[86]

1.8.2.1 Proposition *The form* $(x, y) \in \mathcal{A}^2 \to \mathrm{Tr}(l(x^\alpha y))$ *is a nondegenerate hermitian form on* $\mathcal{A}$ *associated with the involution* α *of* $\mathcal{A}$*, and the signature of the hermitian associated quadratic form on* $\mathcal{A} : x \to \mathrm{Tr}(l(x^\alpha y))$ *is* $(p^2 + q^2, 2pq)$.

1.8.3 Pseudo-Hermitian Structures on the Spaces of Spinors S for $C^+_{r,s}$ ($r - s \equiv \pm 2 \pmod 8$)

Let us take for involution τ the principal antiautomorphism of $\mathcal{A} = C^+_{r,s}$.

The corresponding hermitian form on $C^+_{r,s}$ is $(x, y) \to \mathrm{Tr}(l(x^\alpha y))$. The minimal module of $C^+_{r,s}$ is the space of spinors associated with $C^+_{r,s}$. As usual,[87] let us take u in $V = E_{r,s}$ such that $(u \mid u) = \pm 1$ and put $W = u^\perp$. $C^+_{r,s}$ is the complexification of its subalgebra $C^+(W)$, which is invariant by τ. Since $m = 2k = r + s$, $\dim_{\mathbf{C}} C^+_{r,s} = \dim_{\mathbf{R}} C^+(W) = 2^{2k-2}$ and $\dim_{\mathbf{C}} S = 2^{k-1}$.

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[85] For example, R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., p. 222. One can show it immediately that if ${}^t\bar{U} = -U$, let us take $U_1 = iU$, then we have ${}^t\bar{U}_1 = U_1$.

[86] Cf. also R. Deheuvels, *Groupes Conformes et Algèbres de Clifford*, op. cit., pp. 219–220.

[87] Cf. for example, R. Deheuvels, ibid., pp. 220–221.

The basis of $C^+(W)$ associated with an orthonormal basis $\{e_1, \ldots, e_{2k-1}\}$ of W is a complex basis of $C^+(E_{r,s})$. If we denote by $e_I = e_{i_1} \cdots e_{i_{2k}}$ and $e_L = e_{j_1} \cdots e_{j_{2l}}$ two elements of such a basis, then $e_I^\tau e_L$ is also an element of this basis and a non-scalar one if $I \neq L$. In such a case the translation $l(e_I^\tau e_L)$ permutes the vectors of the basis without any fixed element and with trace equal to zero.

The basis of $C^+(W)$ is a complex orthogonal basis of $C^+_{r,s}$ for the form associated with τ. $(e_I^\tau e_I)$ is a scalar equal to $N(e_I) = (e_{i_1} \mid e_{i_1}) \cdots (e_{i_{2k}} \mid e_{i_{2k}})$ and $\mathrm{Tr}_{\mathbf{C}}(l(e_I^\tau e_I)) = \dim C^+(W).N(e_I)$ is positive if and only if e_I contains an even number of negative vectors of the basis of W and consequently, an even number of positive vectors of the basis of W. As above, if we denote by $p(n)$ and $i(n)$ the respective numbers of subsets of $\{1, 2, \ldots, n\}$ with an even cardinality, respectively an odd cardinality, we have $p(n) = i(n) = 2^{n-1}$. If the corresponding signature of W is (p_1, p_2) according to the choice of u, it is $(r, s-1)$ or $(s, r-1)$, the number of positive vectors of the basis of $C^+(W)$ is $p(p_1)p(p_2)$, and the number of negative vectors is $i(p_1)i(p_2)$. Both these both numbers are equal, and the pseudo-hermitian form $(x, y) \to \mathrm{Tr}(l(x^\tau y))$ on $C^+_{r,s}$ is a neutral one.

Therefore, we deduce that if k is even, ib is a pseudo-hermitian neutral scalar product of signature $(2^{\frac{m}{2}-2}, 2^{\frac{m}{2}-2})$ and then b is a nondegenerate skew-hermitian form of maximal index $2^{\frac{m}{2}-2}$ and if k is odd, S is provided with a pseudo-hermitian scalar product of signature $(2^{\frac{m}{2}-2}, 2^{\frac{m}{2}-2})$.

Since the pseudounitary group of automorphisms of S that leave invariant each of these sesquilinear forms consists in each case of elements u of $C^+_{r,s} \simeq \mathcal{L}_{\mathbf{C}}(S)$ such that $u^\tau u = 1$, Spin $E_{r,s}$ is then embedded into it.

As above, using the lemma given in the appendix, we can verify that

- if k is even, Spin $E_{r,s}$ is embedded into $SU(p, p)$ with $p = 2^{\frac{m}{2}-2}$,
- if k is odd, Spin $E_{r,s}$ is embedded into $SO^*(2p)$ with $p = 2^{\frac{m}{2}-2}$ and then embedded into $SU(p, p)$ with $p = 2^{\frac{m}{2}-2}$.

Then, we have obtained the following result:

1.8.3.1 Theorem *For $m \geq 6$, $m = 2k = r + s$, $r - s \equiv \pm 2 \pmod 8$, the space S of spinors associated with the even algebras $C^+_{r,s}$ possesses a natural complex structure and a pseudo-hermitian neutral scalar product. In each case (k even and k odd)* $\mathrm{Spin}(E_{r,s})$ *is embedded into $SU(p, p)$ with $p = 2^{\frac{m}{2}-2}$.*

We find directly the results obtained by René Deheuvels[88] for $m = 4k + 2$, $r - s \equiv 2 \pmod 4$.

1.8.4 Embedding of the Corresponding Projective Quadric $\tilde{Q}(E_{r,s})$

According to Porteous,[89] the Grassmannian of maximal totally isotropic subspaces of dimension $\left(\frac{1}{2}\right)\dim S$ of the complex space S with $\dim_{\mathbf{C}} S = 2^{\frac{m}{2}-1}$ is homeomorphic

[88] R. Deheuvels, *Groupes Conformes et Algèbres de Clifford*, op. cit.

[89] I. R. Porteous, *Topological Geometry*, op. cit., Theorem 12-12 p. 233 and Proposition 17-46 p. 350.

to $U(2^{\frac{m}{2}-2})$ in each case, k even or k odd. We have then obtained the following theorem:

1.8.4.1 Theorem *For $m = 2k$, $m \geq 6$, $r - s \equiv \pm 2 \pmod 8$, the projective quadric $\tilde{Q}(E_{r,s})$ is embedded into the group $U(p)$, where $p = 2^{\frac{m}{2}-2} = 2^{k-2}$.*

Such a result generalizes the results of Deheuvels.[90] The set of maximal and strictly positive subspaces of S and then of dimension $(\frac{1}{2})\dim S$ is an open set of the Grassmannian $G(S, (\frac{1}{2})\dim S)$ called the semi-Grassmannian of S and denoted by $G^+(S)$. $G^+(S)$ is the classical symmetric hermitian space of type AIII in Elie Cartan's list,[91] $SU(p, p)/S(U(p) \times U(p))$ with $p = 2^{\frac{m}{2}-2}$ and $\tilde{Q}(E_{r,s})$ is embedded into the boundary of $G^+(S)$ into $G(S, (\frac{1}{2})\dim S)$.

As in Deheuvels,[92] $G^+(S)$ can be identified with the symmetric space of involutions of $C^+_{r,s}$ that commute with τ and that are strictly positive.

1.8.5 Concluding Remarks

We can now give the following summary concerning the spin groups Spin $E_{r,s}$:

1.8.5.1 Theorem *In each case $r - s \equiv \pm 3, \pm 1, 0, 4, \pm 2 \pmod 8$, where $m = \dim_{\mathbf{R}} E_{r,s} = r + s \geq 8$, the spin group Spin $E_{r,s}$ is naturally embedded into a pseudounitary neutral group $SU(2^{a(r,s)}, 2^{a(r,s)})$ with*

$$a(r,s) = \begin{cases} \dfrac{m-1}{2} - 1 & \text{if } r - s \equiv \pm 3 \pmod 8, \\ \dfrac{m-1}{2} - 2 & \text{if } r - s \equiv \pm 1 \pmod 8, \\ \dfrac{m}{2} - 1 & \text{if } r - s \equiv 0 \pmod 8, \\ \dfrac{m}{2} - 1 & \text{if } r - s \equiv 4 \pmod 8, \\ \dfrac{m}{2} - 2 & \text{if } r - s \equiv \pm 2 \pmod 8. \end{cases}$$

1.9 Appendix

Proof of the following lemma that has been used before, in particular in 1.5.4.1. $E_{r,s}$ denotes $\mathbf{R}^{r+s}$ endowed with a quadratic form q of signature (r, s); $B(x, y)$ denotes the symmetric bilinear associated form.

[90] R. Deheuvels, *Groupes Conformes et Algèbres de Clifford*, op. cit., part 10.
[91] Cf., for example, S. Helgason, op. cit., p. 354.
[92] R. Deheuvels, *Groupes Conformes et Algèbres de Clifford* op. cit., pp. 224–225.

Lemma *Let* (V, q) *be the quadratic standard pseudo-Euclidean space* $E_{r,s}$ *with* $r \geq 2$. *For any pair of linearly independent vectors* $\{u_1, u_2\}$ *such that* $B(u_1, u_1) = B(u_2, u_2) = -1$, *there exists* $z \in V$ *such that* $B(z, z) = 1$ *and* $B(z, u_1) = B(z, u_2) = 0$.

A quadratic real plane P that inherits two linearly independent vectors u_1 and u_2 with $q(u_1) = q(u_2) = -1$ is necessarily isomorphic to one of the three following standard planes (R^2, q): $P_1 : q_1(x) = -(x_1)^2$; $E_{0,2} : q_2(x) = -(x^1)^2 - (x^2)^2$; $E_{1,1} : q_3(x) = (x^1)^2 - (x^2)^2$.

In the last two cases P is a direct factor of $E_{r,s}$ and $E_{r,s} = P \oplus P^{\perp}$. According to the classical Witt's isomorphism theorem, we have:

- if $P \simeq E_{0,2}$, $P^{\perp} \simeq E_{r,s-2}$ with $r \geq 1$,
- if $P \simeq E_{1,1}$, $P^{\perp} \simeq E_{r-1,s-1}$. In the first case and in the second one, if $r \geq 2$, there always exists $z \in P^{\perp}$ with $q(z) = 1$,
- if $P \simeq P_1$ and if D is the isotropic line of P, there exists an isotropic line D', linearly independent of P, such that $D \perp D'$ and so $P \oplus D'$ is regular and isomorphic to $E_{1,2}$. According to Witt's theorem, $(P \oplus D')^{\perp} \simeq E_{r-1,s-2}$. We obtain the existence of z if $r \geq 2$.

1.10 Exercises

(I) **In the first exercise we summarize classical properties of H**

(A) Let **H** be the standard Clifford algebra $C(E_{0,2}) = \frac{\langle -1,-1 \rangle}{\mathbf{R}}$. Let e_1 and e_2 be an orthogonal basis of $E_{0,2}$ with $e_1^2 = -1 = q(e_1)$, $e_2^2 = -1 = q(e_2)$.

(a) Show that $C(E_{0,2})$ admits the following basis: $1, e_1, e_2, e_1e_2$ with $e_1e_2 + e_2e_1 = 0$. We put $e_1 = i$, $e_2 = j$, $e_1e_2 = k$. Then $\mathbf{H} = \{q = 1.a + ib + jc + kd$ with $a, b, c, d \in \mathbf{R}\}$ is a skew field with the usual addition and the following table for the "unit elements" i, j, k: $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.

(b) If $q = a + ib + jc + kd$, prove that $\pi(q) = a - ib - jc + kd$, $\tau(q) = a + ib + jc - kd$, $\nu(q) = q^* = a - ib - jc - kd$, $\nu(q)$ is called the conjugate quaternion of q.

(c) Prove that for any q in **H**, $\pi(q) = kqk^{-1}$, where $k^{-1} = -k$. More generally for any *even-dimensional* regular standard space (E, q) over the field K show that in the corresponding Clifford algebra $C(E, q)$ for any a in $C(E, q)$, $\pi(a) = uau^{-1}$, where $u = e_1 \cdots e_n$ is the product of the elements of an orthogonal basis for q of the space (E, q).

(d) We put $N(q) = qq^* = q^*q$. Verify that $N(q) = a^2 + b^2 + c^2 + d^2$ and that $N(q)N(q') = N(qq')$ (N is called the usual quaternionic norm). Write the result known as the Euler–Lagrange identity. For $q = a + ib + jc + kd$, $q' = a' + ib' + jc' + kd'$: $(a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2) = (aa' + bb' + cc' + dd')^2 + (ab' - ba' + cd' - dc')^2 + (ac' - bd' - ca' + db')^2 + (ad' + bc' - cb' - da')^2$.

(e) For any q in **H**, we put $q = a + ib + jc + kd = (a + ib) + j(c - id) = u + jv$ with $u = a + ib$, $v = c - id$. We embed the classical field **C** of complex numbers into

H by putting $a + ib \to a + ib + j.0 + k.0 \in \mathbf{H}$. **C** operates by right multiplication into **H** and $1, j$ become the elements of a basis of **H** over **C**. If $z = a + ib \in \mathbf{C}$, verify that $jz = \bar{z}j$, where $\bar{z} = a - ib$ is the classical conjugate complex of z, and that $q^* = a - ib - jc - kd = \bar{u} - jv$.

(f) Show that the form $(q_1, q_2) \in \mathbf{H}^2 \to q_1^* q_2$ is a quaternionic scalar product on **H**. What are the complex components in the basis $\{1, j\}$ of **H** over **C** of this form? Prove now that $SpU(1) = U(2) \cap Sp(2, \mathbf{C})$.

(B) (a) We associate with any quaternion q in **H** a mapping R_q from **H** by $R_q(q') = qq'$.

(b) Show that R_q is an endomorphism of **H** with respect to its structure of a right space over **C**.

(c) Let A_q be the matrix of R_q corresponding to the basis $\{1, j\}$ of **H** over **C**. Determine A_1, A_i, A_j, A_k, and for any $q = u + jv \in \mathbf{H}$ show that

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A_j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A_k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

and that

$$A_{u+jv} = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix}.$$

(d) Verify that the mapping $q \to R_q$ is a representation of the algebra **H** by square matrices of degree 2 with coefficients in **C**.

(e) Show that q^* is associated with the matrix $A_{q^*} = {}^t\bar{A}_q$.

(C) Let $M_n(\mathbf{R})$ be the real associative algebra of square matrices of degree n with coefficients in **R** and let I_n denote the unit element. We assume that there exist two matrices A and B in $M_n(\mathbf{R})$ such that $A^2 = -I_n$, $B^2 = -I_n$, $AB + BA = 0$ (I).

(a) Show that n cannot be odd.

(b) Show that the subspace **H** generated by I_n, A, B, and AB constitutes a subalgebra of $M_n(\mathbf{R})$.

(c) If t, x, y, z are in **R**, determine the product $(tI_n + xA + yB + zAB)(tI_n - xA - yB - zAB)$.

(d) Deduce that I_n, A, B, AB are linearly independent and form a basis of **H** and that **H** is a noncommutative field.

(e) Now, once and for all, we put $n = 4$, $\mathcal{J} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ and 0 the null matrix in $M_2(\mathbf{R})$. We define $A = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & -\mathcal{J} \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}$ in $M_4(\mathbf{R})$ and $C = AB$.

(α) Show that A and B satisfy the above condition (I).

H again denotes the subspace of $M_4(\mathbf{R})$ generated by $I_n, A, B, C = AB$. Its elements are called quaternions. The basis $\{I_4, A, B, C\}$ of **H** is denoted by $\mathcal{B}$.

(β) If M belongs to $\mathbf{H}^*$, show that ${}^tM \in \mathbf{H}$. What is the relation between M^{-1} and ${}^tM$?

(D) Study of the automorphisms of the algebra $\mathbf{H}$.

(a) By definition a pure quaternion is any element M in $\mathbf{H}$ such that $M =^t M$. Show that the set of pure quaternions $\mathcal{L}$ is a space over $\mathbf{R}$ with $\dim_{\mathbf{R}} \mathcal{L} = 3$ and with basis $\{A, B, C\} = \mathcal{C}$. Is $\mathcal{L}$ a subalgebra of $\mathbf{H}$?

(b) $\mathcal{L}$ is provided with the usual structure of a Euclidean space such that $\mathcal{C} = \{A, B, C\}$ is an orthonormal basis. We denote by $(M|N)$ the usual scalar product of M and N in $\mathcal{L}$. $\|M\|$ denotes the corresponding norm. Show that $\frac{1}{2}(MN + NM) = -(M|N)I_4$.

(c) Verify that a quaternion is pure if and only if its square is a square matrix λI_4 with λ a negative real number.

(d) Let ϕ be an isomorphism of algebras from $\mathbf{H}$ into itself. Show that for any $M \in \mathcal{L}$, $\phi(M) \in \mathcal{L}$ with $\|M\| = \|\phi(M)\|$ and that the restriction of ϕ to $\mathcal{L}$ is an orthogonal transformation.

(e) Let M and N be both pure quaternions. We want to show that if $\|M\| = \|N\|$, there exists $P \in \mathbf{H}^*$ such that $M = P^{-1}NP$.

(α) First, study the case that M and N are proportional.

(β) Now we assume that M and N are not proportional. Verify that if $\|M\| = \|N\|$ we have $M(MN) - (MN)N = \|M\|^2(M - N)$. Deduce that there exists a nonzero matrix P such that $MP = PN$.

(f) Now show that if we put $P = \alpha I_4 + Q$, with α in $\mathbf{R}$ and Q in $\mathcal{L}$, Q is orthogonal both to M and N.

(g) Deduce that any algebra isomorphism ϕ from $\mathbf{H}$ into itself is defined by $\phi(M) = P^{-1}MP$, where P is a nonzero element in $\mathbf{H}$. We may observe that such an isomorphism ϕ is determined by $\phi(A)$ and $\phi(B)$ and begin by searching the isomorphisms that leave A invariant.

(h) What is the general theorem that we have verified?

(II) **The construction of Brauer and Weyl**[93]

(a) Let (E, q) be a quadratic regular space over $K = \mathbf{R}$ or $\mathbf{C}$ and let $\wedge E$ be the exterior algebra of E. Let g be the bilinear symmetric form associated with q. Show that with any x in E we can associate an antiderivation d_x of degree -1 of $\wedge E$, with square equal to zero such that for any decomposable p-vector $y_1 \wedge \cdots \wedge y_p$ we have $d_x(y_1 \wedge y_2 \wedge \cdots \wedge y_p) = \sum_{i=1}^{p}(-1)^{i-1} g(x, y_i) y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_p$, where the symbol ˆ means that the corresponding y_i is missing.

(b) We put $L_x : t \in \wedge(E) \to L_x(t) = x \wedge t \in \wedge(E)$. Verify that $L_x^2 = 0$ and that $\varphi : x \in E \to \varphi(x) = d_x + L_x \in \mathcal{L}(\wedge(E))$ is a Clifford mapping from E into $\mathcal{L}(\wedge(E))$.

(c) Deduce that any quadratic regular n-dimensional space over $K = \mathbf{R}$ or $\mathbf{C}$ possesses a Clifford algebra $C(E, q)$ defined as the quotient of $\mathcal{C}(E)$, the tensor algebra of

93 R. Brauer and H. Weyl, Spinors in n dimensions, *Amer. J. Math.*, 57, pp. 425–449, 1935.

E, by the two-sided ideal $N(q)$ generated by the elements $x \otimes x - q(x).1$, $x \in E$, and that $\dim_K C(E, q) = 2^n$. One may prove that the 2^n elements $1_C, e_I = e_{i_1} e_{i_2} \cdots e_{i_p}$ $(1 \leq i_1 < i_2 < \cdots < i_p \leq n)$ constitute a basis of $C(E, q)$ and prove the result by a recurrence using 2(a).

(d) Show that the mapping φ from $E_{3,0} = E_3$ into the real algebra $m(2, \mathbf{C})$ defined for $m = xe_1 + ye_2 + ze_3$ by $\varphi(m) = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis of E_3, is a Clifford mapping. Deduce that $\varphi(E_3)$ is the real space of hermitian matrices with trace equal to zero. We put

$$\sigma_1 = \varphi(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \varphi(e_2) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \varphi(e_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_0 = I$$

(Pauli's matrices). Show that the eight matrices σ_0, σ_1, σ_2, σ_3, $\sigma_1\sigma_2$, $\sigma_1\sigma_3$, $\sigma_2\sigma_3$, $\sigma_1\sigma_2\sigma_3$ are linearly independent over $\mathbf{R}$ and that the real algebra $m(2, \mathbf{C})$ is a Clifford algebra of E_3.

(e) More generally, let $\mathcal{E}_n$ be the standard complex n-dimensional space provided with the standard quadratic form q such that for any x in $\mathcal{E}_n$,

$$q(x) = \sum_{j=1}^{n} (x^j)^2$$

with respect to an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathcal{E}_n$.

(α) First, we assume that n is even, $n = 2r$. Let $m(2^r, \mathbf{C})$ be identified with the tensor product of r copies of $m(2, \mathbf{C})$, i.e., $m(2^r, \mathbf{C}) = m(2, \mathbf{C}) \otimes \cdots \otimes m(2, \mathbf{C})$. We define a mapping p from $\mathcal{E}_n$ into $m(2^r, \mathbf{C})$. For $1 \leq j \leq r$ we put

$$p(e_j) = p_j = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{j-1} \otimes \sigma_1 \underbrace{\otimes I \otimes \cdots \otimes I}_{r-1},$$
$$p(e_{r+j}) = p_{r+j} = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{j-1} \otimes \sigma_2 \underbrace{\otimes I \otimes \cdots \otimes I}_{r-1},$$

and if n is odd $(n = 2r + 1)$,

$$p(e_{2r+1}) = p_{2r+1} = \underbrace{\sigma_3 \otimes \sigma_3 \otimes \cdots \otimes \sigma_3}_{r}.$$

Show that p is a Clifford mapping. Deduce that if n is even, $n = 2r$, $C(\mathcal{E}_{2r}) = m(2^r, \mathbf{C})$, and if n is odd, $n = 2r + 1$, $C(\mathcal{E}_{2r+1}) = m(2^r, \mathbf{C}) \oplus m(2^r, \mathbf{C})$.

(III) **Classical spin groups. Spinors in the usual standard Euclidean space E_3**

(A) Show that Spin $1 = \{1\}$.

(B) Show that Spin $2 \simeq SO(2) \simeq S^1 \simeq U(1)$.

(C) (1) Use the fundamental table to show that $C(E_3) = C_{3,0} \simeq m(2, \mathbf{C})$, where $E_3 = E_{3,0}$, and that the corresponding spinor complex space S is $\mathbf{C}^2$.

(2) Let $\mathbf{H}_1$ be the vector space of pure quaternions, i.e., by definition, of quaternions q such that $\nu(q) = q^* = -q$, where ν is the standard conjugation in $\mathbf{H}$ (cf. exercise I(A)). We identify $\mathbf{H}_1$ with the standard Euclidean space E_3, and S^3 with the multiplicative group of unitary pure quaternions (we recall that as usual, the set of unit vectors $a \in E_{n,0} = E_n = \mathbf{R}^n$ is the unit sphere S^{n-1}) and that a unitary quaternion q is a quaternion such that $N(q) = 1$ (cf. exercise I(A)). If $q \in S^3$ and $x \in \mathbf{H}_1$, show that $(qxq^{-1})^* = -qxq^{-1}$ and thus that $qxq^{-1} \in \mathbf{H}_1$.

(3) Show that the mapping $\varphi : q \to \varphi(q)$ such that $\varphi(q) \cdot x = -qxq^{-1}$ is a homomorphism from S^3 onto $SO(3)$ such that $\operatorname{Ker} \varphi \simeq \mathbf{Z}_2$ and realizes a twofold universal covering. Conclude that the Poincaré group of $SO(3)$ is of order 2.

(4) Deduce that Spin $3 \simeq S^3 \simeq SpU(1)$.

(5) (a) Show that any matrix in $SU(2)$ can be written as

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \text{ or } \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} = A_1$$

with $|a|^2 + |b|^2 = 1$ or respectively $|u|^2 + |v|^2 = 1$ and that the mapping from $SU(2)$ into S^3 defined by $A(a, b) \to a + jb$, respectively $A_1(u, v) \to u + jv$, is an isomorphism (cf. exercise I(B)). Deduce that Spin $3 \simeq SU(2)$ and that the space S of spinors for E_3 can be identified with C^2 and is provided with the standard hermitian scalar product defined for $s = \xi\varepsilon_1 + \eta\varepsilon_2$ and $s' = \xi'\varepsilon_1 + \eta'\varepsilon_2$ by $(s|s') = \xi\bar{\xi}' + \eta\bar{\eta}'$, where $\{\varepsilon_1, \varepsilon_2\}$ is an orthonormal basis for $(\,|\,)$.

(b) Let $\mathbf{C}P_1 = P(C^2)$ be the standard complex projective line. Show that $\mathbf{C}P_1$ can be identified with the Cauchy plane $\tilde{\mathbf{C}} = \mathbf{C}U\{\infty\}$. (If the vector (ξ, η) represents a chosen complex line in $\mathbf{C}^2$, cut it with the affine complex line $\xi = 1$ and put $z = \eta/\xi$ if $\xi \neq 0$, and if we consider the line $\xi = 0$, put $z = \infty$.

(c) Show that the group of homographies of the Cauchy plane $PL(1, \mathbf{C}) = GL(2, \mathbf{C})/\mathbf{C}I = SL(2, \mathbf{C})/\{I, -I\}$. The subgroup $PU(1, \mathbf{C}) = SU(2)/\{I, -I\} = SO(3)$ is the projective unitary group $\mathbf{C}P_1$. Hints: Show that to any linear mapping m from S, identified with $\mathbf{C}^2$, into S there corresponds a homography $p_1(m)$ in the Cauchy plane. Let $\xi' = a\xi + b\eta$, $\eta' = c\xi + d\eta$ and, therefore $\zeta' = \frac{\eta'}{\xi'} = p(m)(\zeta) = \frac{d\zeta + c}{b\zeta + a}$. Give conclusions. Study the converse.

(d) With any $x = x^1 e_1 + x^2 e_2 + x^3 e_3$ in E_3, $x \neq 0$, where $\{e_1, e_2, e_3\}$ is an orthonormal basis, we associate $X = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}$. We recall, (II, d) above that $\varphi : x \to X$ is a Clifford mapping such that $m(2, \mathbf{C})$ is a Clifford algebra of E_3 and that $\varphi(E_3)$ is the real space of hermitian matrices with trace equal to zero.

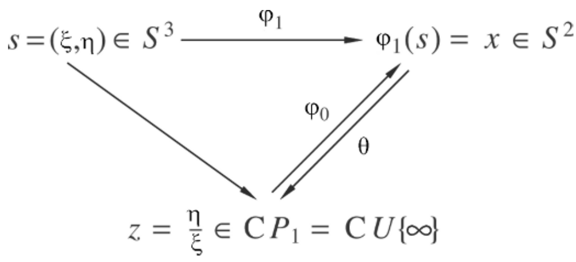
Show that X is unitary if and only if $\|x\| = 1$ and that the eigenvalues of X are $\pm\|x\|$ and that if $\|x\| = 1$, X represents an orthogonal symmetry of S. Verify that to any $x \neq 0$, $x \in E_3$ we can associate a complex line $\Theta(x)$ in S the eigencomplex line associated with the eigenvalue $\|x\|$ of X. Put $\Theta(x) = (\xi, \eta) \in \mathbf{C}^2$. Show that for any $x \in S^2$ — ($\|x\| = 1$) — we have $\frac{\xi}{1+x^3} = \frac{\eta}{x^1+ix^2}$ or equivalently $\frac{\xi}{x^1-ix^2} = \frac{\eta}{1-x^3}$. Verify that Θ is a mapping from $E_3 - \{0\}$ onto $\mathbf{C}P_1$ (and in particular, from S^2 onto $\mathbf{C}P_1$). Show that for any $x \in S^2$, $\Theta(-x)$ is a complex line orthogonal to $\Theta(x)$. Show that Θ can be written $\Theta(x) = z = \frac{\eta}{\xi} = \frac{x^1+ix^2}{1+x^3} = \frac{1-x^3}{x^1-ix^2}$ and that Θ is the classical inversion with center $B(x^1 = 0, x^2 = 0, x^3 = -1)$ and power 2 that sends S^2 into the equatorial plane (x^1, x^2) and B into the point at infinity ∞.

(e) Let $\{\varepsilon_1, \varepsilon_2\}$ be the orthonormal basis for the hermitian scalar product of $S = \mathbf{C}^2$. Let $s = (\xi, \eta)$ be a unitary spinor of S and $t = (\bar{\eta}, -\bar{\xi})$ the corresponding orthogonal unitary spinor. Let σ be the orthogonal symmetry that leaves s invariant such that $\sigma(t) = -t$. Show that the corresponding matrix relative to $\{\varepsilon_1, \varepsilon_2\}$ of σ is

$$A_s = \begin{pmatrix} \xi\bar{\xi} - \eta\bar{\eta} & 2\xi\bar{\eta} \\ 2\bar{\xi}\eta & -(\xi\bar{\xi} - \eta\bar{\eta}) \end{pmatrix}.$$

(f) Solve the equation $A_s = X$ (cf. part II d above) and determine $x = (x^1, x^2, x^3)$ in E_3 such that $\varphi(x) = X = A_s$. Deduce that there exists a mapping φ_1 from S onto E_3 such that $\varphi(s) = x$, whence $\varphi(S^3) = S^2$. Determine $\varphi^{-1}(x)$ for any $x \in S^2$. (Such a mapping φ_1 from S^3 onto S^2 is called the classical Hopf's fibration, the fibers of which are $\varphi_1^{-1}(x)$ for any $x \in S^2$.) Deduce a mapping φ_0 from $\mathbf{C}P_1$ onto S^2 such that $\varphi_0 \circ \Theta = Id$, and that S^2 is homeomorphic with $\mathbf{C}P_1$.

(g) Deduce the following commutative diagram:



such that if $h \in SU(2) = \mathrm{Spin}\,3$ and if $\tilde{h} = p(h) \in SO(3)$ according to the exact sequence $1 \to \mathbf{Z}_2 \to SU(2) \to^p SO(3) \to 1$, we have for any $s \in S$, $\varphi_1(hs) = \tilde{h}\varphi_1(s)$. Let $h \in SU(2) = \mathrm{Spin}(3)$ and let $p_1(h)$ and $\tilde{h} = p(h)$ be the respective corresponding homography of $\mathbf{C}P_1$ and rotation of $SO(3)$. Show that $\varphi_0(p_1(h))(z) = \tilde{h}\varphi_0(z)$, and that $\Theta(\tilde{h}(x)) = p_1(h)\Theta(x)$. Find again that $PU(1) = SU(2)/\{I, -I\} = SO(3)$.

(D) (1) Let $\xi, \eta \in S^3$. Put $X_{\xi,\eta} : \zeta \to \xi\zeta\bar{\eta}$ for any $\zeta \in \mathbf{H}$. Verify that $X_{\xi,\eta} \in SO(4)$.

(2) Verify that $\mu : S^3 \times S^3 \to SO(4)$ defined by $(\xi, \eta) \to X_{\xi,\eta}$ is a homomorphism with kernel equal to $\{(1, 1), (-1, -1)\}$.

(3) Deduce that Spin 4 is isomorphic to $S^3 \times S^3$.

(IV) **Prove the results given as exercises in 1.5.2**
(V) **Orthogonality preserving transformations**
Complete the following outline of a proof:

This is the adaptation of the theorem by J.M. Arnaudies and H. Fraysse, (Cours de mathematiques, 4 : Algebre bilineaire et. geometrie . Dunod Universite, Paris 1999) for the case of a nondegenerate pseudo-Euclidean bilinear form. While in the Euclidean case it is enough to assume that $u \in \mathrm{Hom}_{\mathbf{R}} E$, in the general case we have to add the further assumption that u is invertible. Using a "projection" on an isotropic vector, find a simple counterexample with a noninvertible u.

Proposition
Let u be an invertible element in $\mathcal{L}(E)$ such that $(x|y) = 0$ implies $(ux|uy) = 0$ for all $x, y \in E$. Then $u^*u = C \in \mathbf{R} \setminus \{0\}$.

Proof.
We will first prove that there exists constant $C \neq 0$ such that $(ux|uy) = C\,(x|y)$ for all nonisotropic vectors $x, y \in E$. Let $y \in E$, $y^2 \neq 0$. Let f_y denote the linear functional $f_y(x) = (ux|uy)$. The set $y^\perp \subset E = \{x \in E : (y|x) = 0\}$ is then a hyperplane,

(Exercise: Prove this statement.)

and f_y is identically zero on $y^\perp$.

(Exercise: Prove this statement.)

Since $x \mapsto (x|y)$ is a nonzero linear functional vanishing on $y^\perp$, there exists $f(y) \in \mathbf{R}$ such that $f_y(x) = f(y)(x|y)$ for all $x \in E$.

(Exercise: Prove this statement.)

We will next show that if $y' \in E$, $(y')^2 \neq 0$, then $f(y') = f(y)$. Let us assume, first, that $(y|y') \neq 0$. Then

$$(uy'|uy) = f_y\,(y'|y) = (uy|uy') = f_{y'}\,(y|y') = f_{y'}\,(y'|y),$$

so that $f_y = f_{y'}$.

(Exercise: Check carefully the above.)

Now, suppose that $(y'|y) = 0$. Since both y^2 and $(y')^2$ are not zero, and since $(y + \lambda y')^2 = y^2 + \lambda (y')^2$, there exists $\lambda > 0$ such that $y'' = y + \lambda y'$ is nonisotropic.

(Exercise: Check carefully the above.)

On the other hand $(y|y'') = y^2 \neq 0$, and also $(y'|y'') = \lambda (y')^2 \neq 0$. Therefore $f_y = f_{y''} = f_{y'}$.

(Exercise: Check carefully the above.)

Let us denote the value of the function $y \mapsto f_y$, which, as we have shown, is constant on all nonisotropic vectors, by C. We thus have $(ux|uy) = C(x|y)$ for all nonisotropic x, y in E. Denote $w = u^*u - C.I$. Then $(x|wy) = 0$ for all nonisotropic x, y, and therefore for all vectors e_i of an orthonormal basis of E. It follows that $w = 0$, or $u^*u = C.I$.

(Exercise: Check all points and fill in the gaps.)

The constant C must be different from zero, otherwise we would have, for an arbitrary $y \in E$, $(ux|y) = (ux|uu^{-1}y) = 0$, which, due to the nondegeneracy of the inner product, would imply $u = 0$.

(Exercise: Check all points and fill in the gaps.)

Note: The last part of the theorem does not hold, in a pseudo-Euclidean case, without the assumption that u is invertible. Indeed, it is enough to consider the case of $E_{1,1}$ with the orthonormal basis e_0, e_1, $(e^0)^2 = -1$, $(e_1)^2 = 1$, and u defined as $ux = (x|e_0)\,e_0$. Then u preserves orthogonality, $u \neq 0$, but $u^*u = 0$.

(Exercise: Check all points and fill in the gaps.)

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2

Real Conformal Spin Structures

The second chapter deals with real conformal (pseudo-Euclidean) spin structures. First, we recall the principal properties of Möbius geometry and we present the study of the standard Euclidean plane. Then, we construct covering groups for the general conformal group $C_n(p,q)$ of a standard real space $E_n(p,q)$. We define a natural injective map that sends all the elements of $E_n(p,q)$ into the isotropic cone of $E_{n+2}(p+1,q+1)$, in order to obtain an algebraic isomorphism of Lie groups between $C_n(p,q)$ and $PO(p+1,q+1)$. The classical conformal pseudo-orthogonal flat geometry is then revealed. Explicit matrix characterizations of the elements of $C_n(p,q)$ are given. Then, we define new groups called conformal spinoriality groups. The study of conformal spin structures on riemannian or pseudo-riemannian manifolds can now be made. The preceding conformal spinoriality groups play an essential part. The links between classical spin structures and conformal spin ones are emphasized. Then we study Cartan and Ehresmann conformal connections, Oguie conformal geodesics, and generalized conformal connections. Finally, we present the Vahlen matrices and exercises.

2.1 Some Historical Remarks

The development of quantum mechanics has emphasized the part played by the representations of either Lie groups in theoretical physics or finite groups in theoretical chemistry. As pointed out by Theo Kahan,[1] “the unique mathematical route for studying the properties concerning the symmetries of a physical system and the meaning of the structure of elementary particles and their associate fields is the theory of groups.”[2] Researchers have been led to associate with any of these particles or with a set of

[1] Theo Kahan, *Théorie des Groupes en Physique Classique et Quantique*, Tome 1, Fascicule 1, pp. V–XV, Dunod, Paris, 1960.

[2] We recall the following successful thought of the french mathematician Henri Poincaré (cf. Theo Kahan, ibidem, p. XIV): “the concept of group preexists in our mind, at least, potentially; it is given to us not as a kind of our sensibility but as a kind of our understanding.”

particles a quantum field the structure of which enables us to know the properties of these corpuscles. Principles of symmetry appear and rule the universe of particles. The presence of symmetries implies that of the group-theoretic point of view. According to a well-known happy thought of Albert Einstein, the theory of groups leads the theoretical physicist to a better understanding of the apparent confusion of things.

The Galilean group governs classical mechanics. In special relativity, the first important group is the inhomogeneous or extended Lorentz group, or Poincaré group, the semidirect product of the homogeneous Lorentz group by the group of the translations of the Minkowski classical space. As claimed by Theo Kahan,[3] "mathematics is no longer a tool but "objects" and particles become the representations of the Lorentz group in the sense that the electron is the Dirac equation."

But in the universe of elementary particles, new phase and gauge groups come and enforce their "symmetries." It is well known that the theory of special relativity was established by A. Einstein in 1905 on the basis of the formalism introduced by M. Faraday in his study of the electromagnetic field and of its fundamental laws and completed by J. C. Maxwell, about 1887, for its mathematical presentation.

As emphasized by S. Sternberg,[4] "in order to elaborate a covariant theory, the search of a correct Lie group G for the building of unitary theories including gravity and electromagnetic field, has been the object of the physicists. The classical theory of the electromagnetic field shows the part played by the conformal group $SO^+(2, 4)$, which leaves invariant Maxwell's equations."

First, H. Bateman[5] and E. Cunningham[6] showed how the equations of the electromagnetic field are invariant not only for the Poincaré group but for the larger one: the conformal group. Elie Cartan himself[7] studied the structure of $SO^+(2, 4)$ and showed by an analysis of the roots that $SU(2, 2)$ is a covering group.

R. Brauer and H. Weyl in 1935,[8] and P. A. M. Dirac in 1936,[9] gave a "projective representation" in a six-dimensional space.

Physicists such as W. A. Hepner, Y. Murai, and I. Segal[10] have used the properties of the Lie algebra $\mathcal{L}SO^+(2, 4) = so^+(2, 4)$. Independently, $SO^+(2, 4)$ appears as a group of Dirac's matrices and as a dynamical group for the hydrogen atom.[11]

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[3] Ibid., pp. V–XV.

[4] S. Sternberg, for example, Lectures on Differential Geometry, Prentice-Hall Mathematics series, second printing, 1965.

[5] H. Bateman, The conformal transformations of a space of four dimensions and their applications to geometrical optics, *J. of London Math. Soc.*, 8, 70, 1908; and The transformation of the Electrodynamical Equations, ibid., 8, 223, 1909.

[6] E. Cunningham, The principle of relativity in electrodynamics and an extension thereof, ibid., 8, 77, 1909.

[7] E. Cartan, *Ann l'ENS*, 31, pp. 263–355, 1914.

[8] R. Brauer and H. Weyl, *Amer. J. Math.*, op. cit., 57–425.

[9] P. A. M. Dirac, *Ann. of Math.*, op. cit., 37–429.

[10] W. A. Hepner, op. cit.; Y. Murai, op. cit.; I. Segal, op. cit.

[11] W. A. Hepner, op. cit.

N. H. Kuiper[12] showed that the conformal group of $\mathbf{R}^n$, $n = p+q > 2$, is induced by the projective group that leaves invariant a quadric in $\mathbf{R}P^{n+1}$. His method used projective coordinates and was a generalization of the classical stereographic projection for three-dimensional spaces and emphasized the part played by the classical Liouville's theorem.[13] Such a theorem has also been generalized for pseudo-Euclidean spaces by J. Haantjes.[14]

Besides, according to Nicolas Bourbaki,[15] A. F. Möbius constructed, in the golden period of geometry, a geometry called Möbius geometry, the links of which to physics had not been understood. The notion of spin structure on a manifold V was introduced by A. Haefliger, who specified an idea from Ehresmann.[16] J. Milnor[17] and A. Lichnerowicz[18] have taken an interest in those structures. In a self-contained way, A. Crumeyrolle[19] has developed the study of vector bundles associated with spin structures, in any dimension and signature. He introduced the general definitions of spin structures on a real paracompact n-dimensional smooth pseudo-riemannian (in particular riemannian) manifold and drew up necessary and sufficient conditions for their existence in a purely geometrical way. More precisely, he defined the notion of spinoriality groups such that the existence of a spin structure on V can be submitted to the reduction of the structure group $O(p, q)$ of "the bundle of orthonormal frames of V," to a spinoriality group, after having been complexified. One of the main guiding principles is that the study of fields over curved spaces is nothing but the consideration of spin-orthogonal, or symplectic, fibrations. According to the same guidance, there appears the problem of the investigation of conformal spin structures, in which the part previously assigned to the group $O(p, q)$ will now be given to the conformal one: $C_n(p, q)$.

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[12] N. H. Kuiper, op. cit.

[13] J. A. Schouten and D. J. Struik, op. cit.

[14] J. Haantjes, op. cit.

[15] N. Bourbaki, *Elements d'Histoire de Mathématiques*, op. cit.

[16] A. Haefliger, Sur l'extension du groupe structural d'un espace fibré, *C. R. A. S. Paris*, 243, 1956, pp. 558–560.

[17] J. Milnor, Spin structure on manifolds, *Enseignement Mathématique, Genève*, 2 série 9, 1963, pp. 198–203.

[18] A. Lichnerowicz, Champs spinoriels et propagateurs en relativité générale, *Bull. Soc. Math. France*, 92, 1964, pp. 11–100; and A. Lichnerowicz, Champ de Dirac, champ du neutrino et transformations C. P. T. sur un espace-temps courbe, *Ann. l'I.H.P., Section A (N.S.)*, 1, 1964, pp. 233–290.

[19] A. Crumeyrolle, Structures spinorielles, *Ann. l'I.H.P., Section A (N.S.)*, 11, 1969, pp. 19–55; A. Crumeyrolle, Groupes de spinorialité, *Ann. l'I.H.P., Section A (N.S.)*, 14, 1971, pp. 309–323; A. Crumeyrolle, Dérivations, formes et opérateurs usuels sur les champs spinoriels des variétés différentiables de dimension paire, *Ann. l'I.H.P., Section A (N.S.)*, 16, 1972, pp. 171–201; and A. Crumeyrolle, Fibrations spinorielles et twisteurs généralisés, *Period. Math. Hungar.*, 6, 1975, pp. 143–171 or Spin fibrations over manifolds and generalized twistors, *Differential geometry* (Proc. Sympos. Pure Math., Vol. 27, Part 1, Stanford Univ., Stanford, Calif., 1973), Amer. Math. Soc., Providence, RI, 1975, pp. 53–67.

2.2 Möbius Geometry

2.2.1 Möbius Geometry: A Summary of Classical Results[20]

2.2.1.1 The Space of Hyperspheres

We introduce $V_n = E_{n,0}$, the standard Euclidean space $\mathbf{R}^n$ provided with the norm $\| \ \|$ such that for any $\xi \in V$, $\|\xi\|^2 = g_{ij}\xi^i\xi^j$. Any hypersphere of V admits an equation

$$X^0 g_{ij}\xi^i\xi^j - 2g_{ij}\xi^i X^j + 2X^{n+1} = 0, \tag{1}$$

where the real numbers $X^0, X^1, \ldots, X^{n+1}$, not all equal to zero, are determined up to a nonzero factor. Such real numbers are called homogeneous coordinates of the hypersphere. When X^0 is different from zero, (1) can be written as

$$g_{ij}\left(\xi^i - \frac{X^i}{X^0}\right)\left(\xi^j - \frac{X^j}{X^0}\right) = \frac{g_{ij}X^iX^j - 2X^0X^{n+1}}{(X^0)^2}. \tag{1'}$$

Thus, any equation of type (1) defines

- *a real hypersphere if* $X^0 \neq 0$ and $g_{ij}X^iX^j - 2X^0X^{n+1} > 0$.
- *a hyperplane if* $X^0 = 0$ and $(X^1, \ldots, X^n) \neq (0, \ldots, 0)$.
- *the hyperplane at infinity, by passing to the projective space, if* $X^0 = \cdots = X^n = 0$ and $X^{n+1} \neq 0$.

Therefore, the hyperspheres—in a general sense—of V_n can be represented as points of the projective space P^{n+1}.[21]

We agree to call an "analytic sphere" any point $(X^0, \ldots, X^{n+1})$ *in* $\mathbf{R}^{n+2} \setminus \{0\}$. *If we define* $\pi : \mathbf{R}^{n+2} \setminus \{0\} \to P^{n+1}$ *the canonical projection, any real hypersphere—in a general sense—of* V_n *can be written:* $\pi(X)$, *where* $X \in \mathbf{R}^{n+2} \setminus \{0\}$ with $q(X) = g_{ij}X^iX^j - 2X^0X^{n+1} > 0$.

We recall that $\pi(X) = \pi(Y)$ is equivalent to the existence of a real number λ, $\lambda \neq 0$, such that $Y = \lambda X$. The bilinear symmetric form associated with the quadratic form q is often called the inner product in $\mathbf{R}^{n+2}$ and is defined for X, Y in $\mathbf{R}^{n+2}$ by $X \cdot Y = g_{ij}X^iX^j - X^0Y^{n+1} - Y^0X^{n+1}$, and it is also called the inner product of the corresponding analytic spheres.

We obtain the following results, *which are given as exercises* (cf. below 2.13):

2.2.1.1.1 Proposition *The radius of the hypersphere* $\pi(X)$ *is* $[q(X)/(X^0)^2]^{1/2}$. *The angle* θ *between two intersecting real hyperspheres* $\pi(X)$ *and* $\pi(Y)$ *is determined by* $0 \leq \theta \leq \pi/2$ *and* $\cos\theta = |X \cdot Y|/[(X \cdot X)^{1/2}(Y \cdot Y)^{1/2}]$ *with* $q(X) > 0$ *and* $q(Y) > 0$.

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[20] Cf., for example, M. Berger, *Géométrie*, volume 5, Cedic, Nathan, Paris, pp. 75–80, 1977; A. Toure, Thèse Université Pierre et Marie Curie, Paris VI, 1981; and P. Anglès, Les structures spinorielles conformes réelles, Thèse, Université Paul Sabatier, 1983.

[21] By definition, we recall that $\mathbf{P}^{n+1} = \mathbf{RP}^{n+1}$ denotes the projective space $P(\mathbf{R}^{n+2})$.

2.2.1.1.2 Proposition *If $q(X) = 0$ and $X^0 \neq 0$, $\pi(X)$ is reduced to the point of coordinates $\xi^i = X^i/X^0$, $1 \leq i \leq n$, of V and if $q(Y) > 0$, $X \cdot Y = 0$ is equivalent to "the hypersphere $\pi(Y)$ contains the point ξ."*

2.2.1.2 The Möbius Space

According to what has been said before, there exists a bijective mapping from V_n onto the subset of P^{n+1} consisting of points whose homogeneous coordinates satisfy $q(X) = 0$ and $X^0 \neq 0$.

2.2.1.2.1 Definition (Cf. 1.4.3.2 for general definitions for pseudo-Euclidean spaces) By definition, the Möbius space of order n is the quadric hypersurface Q^n of $\mathbf{P}^{n+1}$ with the homogeneous equation

$$g_{ij} X^i X^j - 2X^0 X^{n+1} = 0.$$

The only point of Q^n whose coordinates satisfy $X^0 = 1$ is the point $(0, 0, \ldots, 0, 1)$ called, by definition, the point at infinity and often denoted by ∞. Thus, usually, one writes X^∞ for X^{n+1}. Q^n can be identified with the one-point compactification of V_n denoted by $\tilde{V}_n = V_n \cup \{\infty\}$.

Let V_{n+1} be the standard Euclidean space $\mathbf{R}^{n+1}$ with the standard norm $\|Y\| = (g_{ij} Y^i Y^j + (Y^{n+1})^2)^{1/2}$ with obvious notation and let V_{n+2} be the standard Euclidean space provided with the norm

$$\|Y\| = (g_{ij} Y^i Y^j + (Y^{n+1})^2 + (Y^0)^2)^{1/2},$$

and let S^n be the unit sphere of V_{n+1}. The stereographic projection s from S^n onto the hyperplane defined by $Y^{n+1} = 0$, in V_{n+1}, with origin $w = (0, 0, \ldots, 0, 1)$ leads to the identification of S^n with $\tilde{V}_n = V_n \cup \{\infty\}$ and $s(w) = \infty$. Let i denote the injective mapping from V_{n+1} into $P(V_{n+2})$, viewed as $V_{n+1} \cup T_\infty$, where T_∞ denotes the hyperplane at infinity $Y^0 = 0$. We obtain that $i(S^n)$ is the projective quadric of $P(V_{n+2})$, an equation of which is, in homogeneous coordinates,

$$g_{ij} Y^i Y^j + (Y^{n+1})^2 - (Y^0)^2 = 0. \tag{2}$$

Let ρ be the projective morphism of $P(V_{n+2})$ induced by the rotation $Y \to r(Y) = Z$ of V_{n+2} defined by $Z^j = Y^j$ for $1 \leq j \leq n$,

$$Z^0 = \frac{X^0 + Y^{n+1}}{\sqrt{2}}, \qquad Z^{n+1} = \frac{Y^0 - Y^{n+1}}{\sqrt{2}}.$$

The image by ρ of the projective quadric defined by (2) is Q^n as $(Y^{n+1})^2 - (Y^0)^2 = -2Z^0 Z^{n+1}$. Thus, we obtain that

$$Q^n = \rho \circ i \circ s^{-1}(\tilde{V}_n). \tag{2'}$$

2.2.1.3 The Möbius Group: $\tilde{M}_n$

2.2.1.3.1 Definitions In the n-dimensional Euclidean space V_n, let A be a point of V_n. For any point P of V_n, let Q the point on the ray AP such that $\overline{AP} \cdot \overline{AQ} = k$, where k is a nonzero real number. By definition, we call the transformation that sends P to Q the inversion of center A and power k. Inversions are not bijective transformations of V_n but of $\tilde{V}_n$.

The Möbius group $\tilde{M}_n$ is defined as the group generated by inversions of $\tilde{V}_n$ and symmetries with respect to a hyperplane.[22] In order to study this group M_n, it is convenient to interpret inversions and symmetries in $\tilde{V}_n$ by means of orthogonal symmetries in the pseudo-Euclidean space of "analytic spheres," i.e., of the space $\mathbf{R}^{n+2}$ provided with its inner product $X \cdot Y = g_{ij} X^i X^j - X^0 Y^{n+1} - Y^0 X^{n+1}$. We will denote such a space by

$$\sum_n = (\mathbf{R}^{n+2}, q).$$

The quadratic form q is of signature $(n+1, 1)$.

Any point $\xi \in \tilde{V}_n$ can be written as $\xi = \pi(X)$, where X is an isotropic element of $\sum_n$ (i.e., such that $q(X) = 0$), where π denotes the projection from $\sum_n$ onto the projective associated space. A linear mapping f from $\sum_n$ into itself induces a punctual transformation of $\tilde{V}_n$ if and only if $q(X) = 0 \Rightarrow q(f(X)) = 0$, for any X of $\sum_n$.

We have the following results left as exercises (cf. below 2.13).

2.2.1.3.2 Proposition *Let $B = (B^0, B^1, \ldots, B^{n+1})$ be a nonisotropic element of $\sum_n$. Let s_B be the associated symmetry with respect to the hyperplane $B^\perp$ defined by $s_B : X \to Y = X - 2(B.B)^{-1}(X.B)B$. Then s_B induces a bijection σ_B of $\tilde{V}_n$ that is*

- *a symmetry with respect to the hyperplane defined by the equation*

$$\sum_{i=1}^{n} B^i \xi^i - B^{n+1} = 0, \quad \textit{if } B^0 = 0.$$

[22] We recall the classic definition, cf., for example, C. Chevalley, *The Algebraic Theory of Spinors*, op. cit., p. 19. Let (E, Q) be a standard regular quadratic n-dimensional space over a field K, of characteristic different from 2. Let $G = O(\mathbf{Q})$ denote the corresponding orthogonal group. Let H be a hyperplane whose conjugate contains a nonsingular vector z. Let $Q(z) = a$. For any $x \in E$, we put $s \cdot x = x - 2a^{-1}B(x,z)z$. An easy computation shows that $Q(s \cdot x) = Q(x)$, i.e., s is orthogonal, and since the conjugate of H is Kz and s does not change if we replace z by kz, $k \neq 0$, s depends only on H and is called the symmetry with respect to the hyperplane H. Moreover, we have the classical following theorem:

Theorem (Cartan–Dieudonné) *Every operation of G belongs to the group G' generated by the symmetries with respect to the hyperplanes whose conjugates contain nonsingular vectors. For the standard Euclidean space V_n, the following result can be written: Every operation of G is a product of symmetries with respect to nonisotropic hyperplanes.*

- *the inversion with center* $B = (B^1/B^0, \ldots, B^n/B^0)$ *and power* $(B_0^2)^{-1} \cdot (B \cdot B)$ *if* $B^0 \neq 0$.

We deduce the following theorem:

2.2.1.3.3 Theorem (Definitions) *The Möbius group* $\tilde{M}_n$ *consists of bijective mapping of* $\tilde{V}_n$ *induced by orthogonal transformations of* $\sum_n = (\mathbf{R}^{n+2}, q)$. $\tilde{M}_n$ *is isomorphic to the classical projective group* $PO(n+1, 1)$. *The pair* $(Q^n, \tilde{M}_n)$ *also denoted later by* M_n, *is called the standard conformal geometry (of type n) or Möbius geometry (of type n);* Q^n *is called the standard* n*-dimensional conformal space also denoted later by* M_n.

2.3 Standard Classical Conformal Plane Geometry[23]

2.3.1 Definition Let M and N be riemannian manifolds. A diffeomorphism $f : M \to N$ is conformal if there exists a differentiable positive function α on M such that for all $x \in M$, for all $a, b \in T_x M$,

$$B(d_x f(a), d_x f(b))_{f(x)} = \alpha(x) B(a, b)_x$$

(i.e., f preserves angles but not necessarily lengths). The set of conformal diffeomorphisms of M onto N is denoted by $\mathrm{Conf}(M, N)$, and in case $M = N$, by $\mathrm{Conf}(M)$. The + superscript will mean that orientation (if any) is preserved.[24]

2.3.2 Definition (Theorem) (Cf. Chapter 2.13, exercise III.) Let $S^2 = \mathbf{C}P^1$ be the Riemann sphere identified with the set $\mathbf{C} \cup \{\infty\}$. One usually defines two classes of mappings from $\mathbf{C}P^1$ onto itself by

$$\text{homographies} \qquad z \to \frac{az+b}{cz+d},$$

$$\text{antihomographies} \qquad z \to \frac{a\bar{z}+b}{c\bar{z}+d},$$

where $\bar{z}$ is the classical conjugate of the complex number z. The group $\mathrm{Conf}(S^2)$ consists of all homographies and antihomographies. $\mathrm{Conf}^+(S^2)$ consists of all homographies. The proof will be given later as an exercise (cf. 2.13).

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23 Cf. Ricardo Benedetti, Carlo Petronio, *Lectures on Hyperbolic Geometry*, Springer, 1992, pp. 7–22.

24 For a riemannian manifold M, we denote by $\mathcal{J}(M)$ the set of all isometric diffeomorphisms of M onto itself, (or isometries of M). If M is supposed to be oriented, $\mathcal{J}^+(M)$ denotes the set of all isometries of M preserving orientation. The differential of a mapping f in a point x of M is denoted by $d_x f$. The scalar product defined on the tangent space $T_x M$ will be denoted by $B(\ ,\ )_x$. We recall that f is an isometry iff for any $x \in M$, for any $v, w \in T_x M$, $B(d_x f(v), d_x f(w))_{f(x)} = B(v, w)_x$.

2.3.3 Proposition (cf. below 2.13) *If we identify* $\mathbf{C}P^1$ *with* $\mathbf{R}^2 \cup \{\infty\}$, *the only elements of* $\mathrm{Conf}\,(\mathbf{C}P^1)$ *are the mappings* $x \to \mu Bi(x)+w$, *where* $\mu > 0$, $B \in O(2)$, *and* i *is either the identity or an inversion and* $w \in \mathbf{R}^2$.

We recall the following classical theorem:

2.3.4 Theorem (Liouville, 1850) (cf. below 2.13 Exercises) *Every conformal diffeomorphism between two domains of* $\mathbf{R}^n$ *has the form* $x \to \mu Bi(x) + w$, *where* $\mu > 0$, $B \in O(n)$, *and* i *is either the identity or an inversion and* $w \in \mathbf{R}^n$.

2.4 Construction of Covering Groups for the Conformal Group $C_n(p, q)$ of a Standard Pseudo-Euclidean Space $E_n(p, q)$[25]

We use the same notations as in 1.4.

2.4.1 Conformal Compactification of Standard Pseudo-Euclidean Spaces $E_n(p, q)$

Let V be a standard pseudo-Euclidean n-dimensional space of type (p, q) (cf. 1.4). We denote by $(\,|\,)$ or $B(\,,\,)$ the associated pseudo-Euclidean scalar product and q the corresponding quadratic form.

We have the following results (cf. below exercises 2.13).

2.4.1.1 Theorem *Let* $H = E_2(1, 1)$ *be the standard hyperbolic real plane provided with an isotropic basis* ε, η *such that* $2(\varepsilon \mid \eta) = 1$ *and let* $E_{n+2}(p + 1, q + 1)$ *be the direct orthogonal sum* $E_n(p, q) \oplus E_2(1, 1) = E_n(p, q) \oplus H = F = E_{n+2}(p + 1, q + 1)$. *The "isotropic" injective mapping* $u : y \to y + q(y)\varepsilon - \eta$ *leads us to identify* $M = P(Q(F)\backslash\{0\})$ *(with the notation of 1.4.3.2), the projective quadric associated with* $E_n(p, q)$ *with the compactified space obtained by the adjunction to* $E_n(p, q)$ *of a projective cone at infinity.* M *is called the Möbius space associated with* $E_n(p, q)$.

M *is identical to the homogeneous space* $PO(F)/\mathrm{Sim}\, V$, *the quotient group of* $PO(F) = O(p + 1, q + 1)/\mathbf{Z}_2$ *by the group* $\mathrm{Sim}\, V$ *of similarities of* V. *Moreover,* $PO(F)$ *is identical to the group* $\mathrm{Conf}\,(E_n(p, q))$ *of conformal transformations of* $E_n(p, q)$ *and is generated by products of affine similarities and inversions according to a theorem of Haantjes that extends to pseudo-Euclidean spaces, the Liouville theorem*[26] *(for* $p + q \geq 3$*).*

[25] (a) P. Anglès, Construction de revêtements du groupe conforme d'un espace vectoriel muni d'une métrique de type (p, q), *Annales de l'I.H.P., section A*, vol XXXIII no 1, 1980, pp. 33–51. (b) R. Deheuvels, Groupes conformes et algèbres de Clifford, *Rend. Sem. Mat. Univers. Politecn. Torino*, vol. 43, 2, 1985, pp. 205–226.

[26] J. Haantjes, Conformal representations of an n-dimensional Euclidean space with a non-definitive fundamental form on itself, *Nedel. Akad. Wetensch. Proc.*, 40, 1937, pp. 700–705.

2.4.2 Covering Groups of Conf $(E_n(p, q)) = C_n(p, q)$[27]

2.4.2.1 Notation

We use the standard notation of Chapter 1 (1.4.1). π denotes the principal automorphism of the standard Clifford algebras $C_{p,q}$; τ denotes the principal antiautomorphism of $C_{p,q}$. We recall the following exact sequences:

$$1 \to \mathbf{Z}_2 \to RO(p, q) \to O(p, q) \to 1,$$

$$1 \to \mathbf{Z}_2 \to RO^+(p, q) \to SO(p, q) \to 1,$$

$$1 \to \mathbf{Z}_2 \to \text{Spin}(p, q) \to SO^+(p, q) = O^{++}(p, q) \to 1, ((p, q) \neq (1, 1)).$$

We put, for any g in G, the Clifford group, and for any x in $E_n(p, q)$, $\varphi(g)x = gxg^{-1}$ and for any g in $\tilde{G}$, the regular Clifford group, and for any x in $E_n(p, q)$, $\psi(g)x = \pi(g)xg^{-1}$.

2.4.2.1.1 Definition Let f be a continuously differentiable mapping from an open set U of $E_n(p, q)$ into $E_n(p, q)$. f is said to be conformal in U if there exists a continuous function λ from U into $\mathbf{R}^* = \mathbf{R}\backslash\{0\}$ such that for almost all $x \in U$ and for all $a, b \in E_n(p, q)$, $B(d_x f(a), d_x f(b)) = \lambda^2(x)B(a, b)$ (where B is the polar bilinear symmetric form associated with the quadratic form q of type (p, q), and where $d_x f$ is defined in footnote 24.)

2.4.2.1.2 Theorem (Definition) *Abusively, one defines the conformal group of $E_n(p, q)$ as the restriction of $PO(p + 1, q + 1)$ to the projective quadric M, or Möbius space M; M is homeomorphic with $S^p \times S^q/\mathbf{Z}_2$ (cf. below exercises).*

Thus, by definition, Conf $(E_n(p, q))$ is the set of all projective transformations that leave M invariant. We emphasize the fact that some such transformations are defined only on open sets of $E_n(p, q)$. The Haantjes theorem[28] allows us to express that the only conformal transformations of $E_n(p, q)$, $n \geq 3$, are the products of affine similarities and inversions.

2.4.2.2 Construction of a Covering Group of $C_n(p, q)$

Let $C_{p+1,q+1}$ be the standard Clifford algebra of $E_{n+2}(p + 1, q + 1) = E_n(p, q) \oplus E_2(1, 1) = E_n(p, q) \oplus H$. Once and for all, we consider $\{e_1, \ldots, e_n\}$ an orthonormal basis of $E_n(p, q)$ with $(e_i)^2 = 1$ for $1 \leq i \leq p$ and $(e_i)^2 = -1$ for $p + 1 \leq i \leq p + q = n$, and $\{e_0, e_{n+1}\}$ an orthonormal basis of the standard hyperbolic plane

[27] P. Anglès, Construction de revêtements du groupe conforme d'un espace vectoriel muni d 'une métrique de type (p, q), *Annales de l'I.H.P., section A, Physique théorique*, vol. XXXIII no. 1, 1980, pp. 33–51.

[28] J. Haantjes, op. cit.

$H = E_2(1,1)$ with $e_0^2 = 1$ and $e_{n+1}^2 = -1$. If we put $x_0 = (e_0 + e_{n+1})/2$ and $y_0 = (e_0 - e_{n+1})/2$, $\{x_0, y_0\}$ is an isotropic basis, or Witt basis, of H with $2B(x_0, y_0) = 1$. At other places, we will use also the notation x_{0^*} or y_{n+1} instead of y_0. If necessary, the definition will be recalled. We are going to construct:

(a) an injective mapping u from $E_n(p,q)$ into the isotropic cone C_{n+2} of $E_{n+2}(p+1, q+1)$ such that any mapping $f \in C_n(p,q)$ can be described by means of elements of $RO(p+1, q+1)$. More precisely, for almost all $x \in E_n(p,q)$ and any f in $C_n(p,q)$ there exist g in $RO(p+1, q+1)$ and $\sigma_g(x)$ in $\mathbf{R}$ such that

$$\psi(g) \cdot u(x) = \pi(g)u(x)g^{-1} = \sigma_g(x)u(f(x)). \qquad \text{(A)}$$

(b) a morphism of groups $\tilde{\varphi}$ with a discrete kernel, from $RO(p+1, q+1)$ onto $C_n(p,q)$: $g \to \tilde{\varphi}(g) = f \in C_n(p,q)$.

2.4.2.2.1 Construction of u

We put $u(x) - x = ae_0 + be_{n+1}$ with $(a,b) \in \mathbf{R}^2$. Since $(u(x))^2$ is equal to zero, we find easily that $x^2 = b^2 - a^2$. We can choose $a = \frac{1}{2}(x^2 - 1) = \frac{1}{2}(q(x) - 1)$ and $b = \frac{1}{2}(1 + x^2) = \frac{1}{2}(1 + q(x))$. Thus

$$u(x) = \frac{1}{2}(x^2 - 1)e_0 + x + \frac{1}{2}(1 + x^2)e_{n+1}, \qquad \text{(B)}$$

or equivalently,

$$u(x) = x^2 x_0 + x - y_0. \qquad \text{(B}_1\text{)}$$

We obtain again the "isotropic" injective mapping already given in Theorem 2.4.1.

2.4.2.2.2 Determination of $\tilde{\varphi}$

We are going to show successively that

(a) there exists a mapping $\tilde{\varphi}$ from $RO(p+1, q+1)$ into the set $\mathcal{E}$ of mappings from $E_n(p,q)$ into itself $g \to \tilde{\varphi}(g) = f$ such that for almost all $x \in E_n(p,q)$ there exists $\sigma_g(x)$ in $\mathbf{R}$ with

$$\psi(g).u(x) = \pi(g)u(x)g^{-1} = \sigma_g(x)u(\tilde{\varphi}(g)(x)) = \sigma_g(x)u(f(x)). \qquad \text{(A)}$$

(b) $\tilde{\varphi}$ is a morphism from the multiplicative group $RO(p+1, q+1)$ into $(\mathcal{E}, \circ)$, where "$\circ$" is the usual composition of mappings.

(c) $\tilde{\varphi}(RO(p+1, q+1))$ is a group that according to the Haantjes theorem, can be identified with the conformal group $C_n(p,q)$.

(d) $A = \operatorname{Ker} \tilde{\varphi}$ is a discrete group.

For convenience, we will use the notation for u given by (B) and for $\sigma_g(x) \neq 0$, we will denote $(\sigma_g(x))^{-1}$ by $\lambda_g(x)$.

Step a. For any g in $RO(p+1, q+1)$, $\psi(g) \in O(p+1, q+1)$. Let us put $\psi(g) \cdot u(x) = X^0 e_0 + X^{n+1} e_{n+1} + X$, $(X^0, X^{n+1}) \in \mathbf{R}^2$, $X \in E_n(p,q)$. (A) is equivalent to

$$\frac{1}{2}\sigma_g(x)(f^2(x)-1)e_0 + \sigma_g(x) f(x) + \frac{1}{2}\sigma_g(x)(f^2(x)+1) = X^0 e_0 + X + X^{n+1} e_{n+1},$$

i.e., with

$$\begin{cases} \frac{1}{2}\sigma_g(x)(f^2(x)-1) = X, \\ \frac{1}{2}\sigma_g(x)(f^2(x)+1) = X^{n+1}, \\ \sigma_g(x) f(x) = X, \end{cases}$$

i.e., with

$$\begin{cases} \sigma_g(X) = X^{n+1} - X^0, \\ \sigma_g(X) f^2(x) = X^{n+1} + X^0 \quad (A_1), \\ \sigma_g(X) f(x) = X. \end{cases}$$

• Thus, for $\sigma_g(x) \neq 0$, to any $g \in RO(p+1, q+1)$, we can associate a mapping f from E_n into E_n, independent of the "writing" of g in $RO(p+1, q+1)$ for $\sigma_g(X) = X^{n-1} - X \neq 0$, $f(x) = X/(X^{n+1} - X^0)$. We set $\tilde{\varphi}(g) = f$.

• If $\tilde{\varphi}(g) = f$ exists, we note that $f(x)$ and $\sigma_g(x)$, also denoted by $\sigma_{g,f}(x)$, are defined without ambiguity.

Assume that for g given in $RO(p+1, q+1)$ there exist two corresponding elements, by $\tilde{\varphi}$: f and f'. For almost all x in $E_n(p,q)$, with obvious notation, we have $\sigma_{g,f}(x) u(f(x)) = \sigma_{g',f'} u(f'(x))$, which implies that $u(f(x))$ and $u(f'(x))$ are proportional for almost all x in $E_n(p,q)$. An easy computation shows that $u(x) = \lambda u(x')$ with $\lambda \in \mathbf{R}^*$ implies that $x = \lambda x'$ and $x^2 = \lambda x'^2$ and then $\lambda^2 x'^2 = \lambda x'^2$, whence $\lambda = 1$ and $x = x'$. Then, for almost all x in $E_n(p,q)$, $f(x) = f'(x)$, and $\sigma_{g,f}(x) = \sigma_{g,f'}(x)$ will be now denoted by $\sigma_g(x)$.

Step b. Put $\tilde{\varphi}(g) = f$ and $\tilde{\varphi}(g') = f'$. It is easy to verify that $\tilde{\varphi}(g'g) = f' \circ f$. Since $\psi(g) \cdot u(x) = \sigma_g(x) u(f(x))$ and $\psi(g') \cdot u(x) = \sigma_{g'}(x) u(f'(x))$, we have

$$\psi(g'g).u(x) = \pi(g')(\pi(g)u(x)g^{-1})g'^{-1} = \pi(g')(\sigma_g(x)u(f(x))g'^{-1}$$

for almost all x in $E_n(p,q)$, and there, $\psi(g'g) \cdot u(x) = \sigma_{g'}(f(x))\sigma_g(x) u(f' \circ f(x))$, whence $\tilde{\varphi}(g'g) = f' \circ f$ and

$$\sigma_{g'g} = \sigma_{g'}(f(x))\sigma_g(x). \qquad \text{(C)}$$

Step c. Using step b and the fact (1.2.2.7) that $RO(p+1, q+1)$ is the multiplicative group consisting of products of vectors x in $E_{n+2}(p+1, q+1)$ such that $x^2 = \pm 1 = N(x)$, we are led to determine $\tilde{\varphi}(v)$, where $v = v_1 + v_2 + v_3 \in E_{n+2}$, with $v_1 = \lambda^1 e_0$, $v_2 = \lambda^{n+1} e_{n+2}$, $v_3 \in E_n(p,q)$, $(\lambda^0, \lambda^{n+1}) \in \mathbf{R}^2$, and $v^2 = \pm 1$. It is well known[29] that

$$\psi(v).u(x) = u(x) - \frac{2B(u(x), v)}{N(v)} v, \quad \text{with } N(v) = v^2 = \varepsilon = \pm 1.$$

We put, as in step a above, $\psi(v).u(x) = X^0 e_0 + X + X^{n+1} e_{n+1}$, $X \in E_n(p, q)$, $(X^0, X^{n+1}) \in \mathbf{R}^2$.
An easy computation leads to

$$X^0 = \varepsilon \Big\{ (\lambda^0)^2 \frac{1}{2}(1 - x^2) - (\lambda^{n+1})^2 \frac{1}{2}(x^2 - 1) + (v^3)^2 \frac{1}{2}(x^2 - 1) + (1 + x^2)\lambda^0 \lambda^{n+1} - 2\lambda^0 B(x, v_3) \Big\}$$

$$X^{n+1} = \varepsilon \Big\{ (\lambda^0)^2 \frac{1}{2}(1 + x^2) + (\lambda^{n+1})^2 \frac{1}{2}(1 + x^2) + (v^3)^2 \frac{1}{2}(x^2 + 1) + (1 - x^2)\lambda^0 \lambda^{n+1} - 2\lambda^{n+1} B(x, v_3) \Big\}$$

$$X = \varepsilon \Big\{ x \underbrace{((\lambda^0)^2 - (\lambda^{n+1})^2 + (v^3)^2)}_{v^2 = \varepsilon} - v_3(2B(x, v_3 - \lambda^0(1 - x^2) - \lambda^{n+1}(1 + x^2)) \Big\}$$

Using the previous system (A_1) given in step a above, we obtain

$$\begin{aligned} \sigma_g(x) = X^{n+1} - X^0 &= \varepsilon\{x^2(\lambda^{n+1} - \lambda^0)^2 \\ &\quad + (v^3)^2 - 2B(x, v_3)(\lambda^{n+1} - \lambda^0)\} = \varepsilon(x(\lambda^{n+1} - \lambda^0) - v_3)^2, \\ \sigma_g(x) f(x) &= \varepsilon\{x v^2 - v_3(2B(x, v_3) + x^2(\lambda^0 - \lambda^{n+1}) - (\lambda^0 + \lambda^{n+1}))\}, \\ \sigma_g(x) f^2(x) &= \varepsilon\{v_3^2 x^2 + (\lambda^0 + \lambda^{n+1})(\lambda^0 + \lambda^{n+1} - 2B(x, v_3))\}. \end{aligned}$$

We consider the "hyperquadric" defined in $E_n(p, q)$ by $\sigma_g(x) = 0$, i.e., $(x(\lambda^{n+1} - \lambda^0) - v_3)^2 = 0$. For $\sigma_g(x) \neq 0$,

$$f(x) = \frac{x v^2 - v_3(2B(x, v_3) + x^2(\lambda^0 - \lambda^{n+1}) - (\lambda^0 + \lambda^{n+1}))}{(x(\lambda^{n+1} - \lambda^0) - v_3)^2}, \tag{D}$$

$\sigma_g(x) = \varepsilon(x(\lambda^{n+1} - \lambda^0) - v_3)^2$, $v^2 = \varepsilon = \pm 1$.

I. $\boldsymbol{\lambda^{n+1} - \lambda^0 = 0}$**.** $v^2 = \varepsilon = v_3^2 + (\lambda^0)^2 - (\lambda^{n+1})^2 = v_3^2$, $\sigma_g(x) = \varepsilon^2 = 1$,

$$f(x) = \frac{\varepsilon x - v_3(2B(x, v_3)) - (\lambda^0 + \lambda^{n+1}))}{\varepsilon} = x - v_3(2B(x, v_3)\varepsilon) + 2\lambda^0 \varepsilon v_3.$$

[29] Cf. for example, R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., chapitre IV.

As usual, $x \to x - 2B(x, v_3)\varepsilon v_3 = u_{v_3}(x)$,[30] where u_{v_3} denotes the orthogonal symmetry with respect to $(v_3)^\perp$.

Since according to the classical Cartan–Dieudonné theorem[31] every element $u \in O(p, q)$ can be expressed as the product of at most $n = p + q$ symmetries with respect to nonisotropic hyperplanes and since $\tilde{\varphi}$ is a homomorphism of groups and $2\lambda^0 \varepsilon v_3 = y$ is in $E_n(p, q)$, we obtain the "generic element" of the classical Poincaré group, the semidirect product of $O(p, q)$ by the group of translations of E_n.

One can verify immediately that for these elements we have $\sigma_g(x) = 1$. In the special case $\lambda^{n+1} = \lambda^0 = 0$, we find that $f(x) = u_{v_3}(x) \in O(p, q)$ with $\sigma_g(x) = 1$.

II. $\boldsymbol{\lambda^{n+1} - \lambda^0 \neq 0}$**.** $v^2 = \varepsilon = (\lambda^0)^2 - (\lambda^{n+1})^2 + v_3^2$. We have $\sigma_g(x) = \varepsilon((\lambda^{n+1} - \lambda^0)x - v_3)^2$,

$$f(x) = \frac{xv^2 + (\lambda^0 + \lambda^{n+1})v_3}{[(\lambda^0 + \lambda^{n+1})x + v_3]^2} - \frac{v_3[2B(x, v_3) + x^2(\lambda^0 - \lambda^{n+1})]}{[(\lambda^0 - \lambda^{n+1})x + v_3]^2},$$

i.e.,

$$f(x) = \frac{[(\lambda^0)^2 - (\lambda^{n+1})^2]x + (\lambda^0 + \lambda^{n+1})v_3}{[(\lambda^0 - \lambda^{n+1})x + v_3]^2} + \frac{v_3^2 x - v_3[2B(x, v_3) + x^2(\lambda^0 - \lambda^{n+1})]}{[(\lambda^0 - \lambda^{n+1})x + v_3]^2}.$$

Then,

$$f(x) = (\lambda^0 + \lambda^{n+1})\frac{[(\lambda^0 - \lambda^{n+1})x + v_3]}{[(\lambda^0 - \lambda^{n+1})x + v_3]^2} + (x + \frac{v_3}{(\lambda^0 - \lambda^{n+1})})v_3^2 . \frac{1}{[(\lambda^0 - \lambda^{n+1})x + v_3]^2} - \frac{v_3[2B(x, v_3)(\lambda^0 - \lambda^{n+1}) + x^2(\lambda^0 - \lambda^{n+1})^2 + v_3^2]}{(\lambda^0 - \lambda^{n+1})[2B(x, v_3)(\lambda^0 - \lambda^{n+1}) + x^2(\lambda^0 - \lambda^{n+1})^2 + v_3^2]},$$

which is equivalent to

$$f(x) = (\lambda^0 + \lambda^{n+1})\frac{1}{[(\lambda^0 - \lambda^{n+1})x + v_3]} + \frac{v_3^2}{(\lambda^0 - \lambda^{n+1})} . \frac{1}{([\lambda^0 - \lambda^{n+1}]x + v_3)} - \frac{v_3}{(\lambda^0 - \lambda^{n+1})},$$

i.e.,

$$f(x) = \frac{1}{(\lambda^0 - \lambda^{n+1})}[(\lambda^0)^2 - (\lambda^{n+1})^2 + v_3^2] . \frac{1}{[(\lambda^0 - \lambda^{n+1})x + v_3]} - \frac{v_3}{[\lambda^0 - \lambda^{n+1}]},$$

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[30] Cf., for example, R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., pp. 350–351 or C. Chevalley, *The Algebraic Theory of Spinors*, op. cit., p. 19 and above, 2.2.1.3.a.

[31] C. Chevalley, *The Algebraic Theory of Spinors*, op. cit., pp. 19–20.

that is,

$$f(x) = \frac{\varepsilon}{(\lambda^0 - \lambda^{n+1})^2} \cdot \frac{1}{(x + \frac{v_3}{\lambda^0 - \lambda^{n+1}})} - \frac{v_3}{(\lambda^0 - \lambda^{n+1})}. \tag{E}$$

This is the general form of f corresponding to a vector $v \in E_{n+2}$. We find inversions with center 0 and power in $\mathbf{R}^*$ (put $v_3 = 0$). Moreover, we note that dilations of E_n correspond to elements of $RO(p+1, q+1)$, which can be written as $\exp((1/2)\eta e_0 e_{n+1})$, $\eta \in \mathbf{R}^*$.

Since we have shown that $\tilde{\varphi}$ is a homomorphism of groups, according to the Haantjes theorem (cf. above 2.4.1), $\tilde{\varphi}(RO(p+1, q+1))$ can be identified with the group denoted above by $C_n(p, q)$. $\tilde{\varphi}(RO(p+1, q+1)) = C_n(p, q)$.

We agree by definition to call $\tilde{\varphi}(RO^+(p+1, q+1)) = (C_n(p, q))_r$ *the restricted conformal group.*

2.4.2.2.3 Remarks on Formula (E)

(a) If $v_3 = 0$ we find that

$$f(x) = \left(\frac{\lambda^0 + \lambda^{n+1}}{\lambda^0 - \lambda^{n+1}}\right) \frac{1}{x} = \frac{\varepsilon}{(\lambda^0 - \lambda^{n+1})^2} \frac{1}{x}.$$

We find the inversions with center O and power in $\mathbf{R}^*$. $(\lambda^0 = 1, \lambda^{n+1} = 0)$ is associated with the inversion with center 0 and power 1 and $(\lambda^0 = 0, \lambda^{n+1} = 1)$ with the inversion with center 0 and power -1, associated respectively with e_0 and e_{n+1}.

(b) If $v_3 \neq 0$ and $v_3^2 = 0$, since $(\lambda^0)^2 - (\lambda^{n+1})^2 = \varepsilon$ is equivalent to

$$\frac{1}{\lambda^0 - \lambda^{n+1}} = \varepsilon(\lambda^0 + \lambda^{n+1}),$$

we find that

$$f(x) = (\lambda^0 + \lambda^{n+1})\left(\frac{1}{x(\lambda^0 - \lambda^{n+1}) + v_3} - \varepsilon v_3\right).$$

(c) We note that the case that v_3 is isotropic and $(v_1 + v_2)$ isotropic cannot occur: otherwise, E_{n+2} would be totally isotropic.

(d) Since, for any x in $E_n(p, q)$, $\sigma_v(x) = \varepsilon(x(\lambda^{n+1} - \lambda^0) - v_3)^2$ for v in E_{n+2}, and since $\sigma_{g'g}(x) = \sigma_{g'}(f(x))\sigma_g(x)$, we deduce that $\sigma_g(x)$ is different from zero when f is defined and that there exists $\lambda_g(x) = (\sigma_g(x))^{-1}$. If $\lambda^{n+1} - \lambda^0 = 0$, $\tilde{\varphi}(v) = f$ is defined for any x in $E_n(p, q)$. If $\lambda^{n+1} - \lambda^0 \neq 0$, $\tilde{\varphi}(v) = f$ is defined for any $x \in E_n(p, q) \backslash \{v_3/(\lambda^{n+1} - \lambda^0)\}$.

2.4.2.2.4 Determination of $\mathcal{A} = \operatorname{Ker} \tilde{\varphi}$

We have that $g \in \mathcal{A} = \operatorname{Ker} \tilde{\varphi}$ iff for all x in $E_n(p, q)$, $\pi(g).u(x)g^{-1} = \psi(g).u(x)$ is proportional to $u(x)$. Searching in the group $C_n(p, q)$ to determine the identity mapping, one can easily verify that for any x in $E_n(p, q)$ and for any g in $\mathcal{A} = \operatorname{Ker} \tilde{\varphi}$, $\sigma_g(x) = \sigma_g(-x) = \sigma_{\mathrm{Id}}(x) = \varepsilon = \pm 1$.

The successive choices of x in $E_n(p, q)$ and $-x$ in $E_n(p, q)$ such that $x^2 = 1$,[32] and $(-x)^2 = 1$, lead us to verify that for any g in $\mathcal{A}$, $\psi(g).e_{n+1} = \varepsilon e_{n+1}$, $\psi(g).x = \varepsilon x$.[33] In the same way, the successive choice of x, $(x^2) = -1$, and of $-x$ such that $(-x)^2 = -1$ leads to $\psi(g).e_0 = \varepsilon e_0$, and $\psi(g).x = \varepsilon x$ $(\forall x)(x^2 = -1)$ $(\forall g)(g \in \mathcal{A})$. Moreover, for any $x : x^2 = 0$, we easily find that $\psi(g).x = \varepsilon x$. Thus, for any z in E_{n+2}, $z = \lambda^0 e_0 + \lambda^{n+1} e_{n+1} + x$, with x in $E_n(p, q)$, $(\lambda^0, \lambda^{n+1}) \in \mathbf{R}^2$, we find that $\psi(g).z = \varepsilon z$, where $\varepsilon = \pm 1$. Thus, $g \in \mathcal{A} \Leftrightarrow \psi(g) = \mathrm{Id}_{E_{n+2}}$ or $\psi(g) = -\mathrm{Id}_{E_{n+2}}$.[34]

Since classically $\psi^{-1}\{\mathrm{Id}\} = \{1, -1\}$ and $\psi(e_0 e_{n+1} e_1 \cdots e_n) = -\mathrm{Id}_{E_n}$ according to the general results of Chapter 1, we obtain that $\operatorname{Ker} \tilde{\varphi} = \{1, -1, e_N, -e_N\}$, where $e_N = e_0 e_{n+1} e_1 \cdots e_n$. $\mathcal{A}$ is discrete. One can easily verify that if respectively $e_N^2 = 1$, respectively $e_N^2 = -1$, $\mathcal{A}$ is isomorphic with $\mathbf{Z}_2 \times \mathbf{Z}_2$ or respectively $\mathbf{Z}_4$.

2.4.2.3 Complements: Table of Results (see below table of results)[35]

We present some results that *will be proved below in exercises* (2.13). We denote by u_i or u_{e_i} the following mapping:

$$u_{(e_i)} : x \to u_{(e_i)}(x) = x - 2B(x, e_i)\mathcal{E}_i e_i \begin{cases} \mathcal{E}_0 = -\mathcal{E}_{n+1} = 1, & \\ \mathcal{E}_i = 1, & 1 \leq i \leq p, \\ \mathcal{E}_i = -1, & p+1 \leq i \leq n. \end{cases}$$

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[32] This is possible according to the following result of Chevalley (*The Algebraic Theory of Spinors*, op. cit., p. 14): Assume that B is nondegenerate and that there is an $x \neq 0$ in E such that $q(x) = 0$. Then for any a in the field K ($K = \mathbf{R}$ or $\mathbf{C}$), there is a in E such that $q(z) = a$.

[33] It is easy to write

$$\begin{cases} x : x^2 = 1 \Rightarrow u(x) = x + e_{n+1} \\ -x : (-x)^2 = 1 \Rightarrow u(-x) = -x + e_{n+1} \end{cases}$$
$$\Rightarrow \begin{cases} \psi(g).(x + e_{n+1}) = \sigma_{\mathrm{Id}}(x)(x + e_{n+1}) \\ \psi(g).(-x + e_{n+1}) = \sigma_{\mathrm{Id}}(-x)(-x + e_{n+1}) \end{cases}$$
$$\Rightarrow \psi(g).e_{n+1} = \varepsilon e_{n+1}$$

and $\psi(g) \cdot x = \varepsilon x$, $(\forall g)(g \in \mathcal{A})$, $(\forall x)(x^2 = 1)$.

[34] Remark: according to a well-known result (cf., for example, E. Artin, *Algèbre Géométrique*, op. cit., Théorème 3.18, p. 126), if E is a quadratic regular space over $K = \mathbf{R}$ or $\mathbf{C}$, if $\sigma \in O(q)$ leaves any isotropic line invariant, $\sigma = \pm \mathrm{Id}_E$. But there, $\psi(g)$ belongs to $O(p+1, q+1)$ and $u(E_n(p, q))$ is included *into* the isotropic cone C_{n+2} of $E_{n+2}(p+1, q+1)$. Our method is different.

[35] Cf. P. Anglès, Construction de revêtements du groupe conforme d'un espace vectoriel muni d'une métrique de type (p, q), *Annales de l'I.H.P., section A*, vol. XXIII no. 1, 1980, pp. 33–51, op. cit.

2.4.2.3.1 Remark In agreement with a result given above in 1.5.2.2.1, the table of results given in 2.4.2.3 shows that the preimages g by $\tilde{\varphi}$ of the elements of the group $GO(p,q)$ of similarities of the space $E(p,q)$—following the notation used by Jean Dieudonné—satisfy the following property: there exists $\mu \in \mathbf{R}^*$ such that $g^\tau g = \mu 1$.

2.4.2.4 Covering Groups of $C_n(p,q)$

(1) We have found that algebraically, $C_n(p,q)$ is isomorphic to $\frac{RO(p+1,q+1)}{\mathcal{A}}$. Classically, $RO(p+1,q+1)$ is provided with the structure of a Lie Group.[36] Let us denote by i the algebraic isomorphism from $\frac{RO(p+1,q+1)}{\mathcal{A}}$ onto $C_n(p,q)$. Since $\mathcal{A}$ is a discrete group, we can postulate that i defines a topological isomorphism between $\frac{RO(p+1,q+1)}{\mathcal{A}}$ and $C_n(p,q)$. Thus $C_n(p,q)$ becomes a topological group and even a Lie group. Since $(e_N)^2 = (e_0 e_{n+1} e_1 \cdots e_n)^2 = (-1)^{r+q}$, with $n = 2r$ or $n = 2r+1$, we have obtained the following result.

2.4.2.4.1 Proposition *If r and q are of the same parity ($r+q$ even, $e_N^2 = 1$), $RO(p+1,q+1)$ is a double twofold covering group of the conformal group $C_n(p,q)$.*

If r and q are of opposite parity ($r+q$ odd, $e_N^2 = -1$), $RO(p+1,q+1)$ is a fourfold covering group of $C_n(p,q)$.

2.4.2.4.2 Fundamental Isomorphism

We know that $\psi(\mathcal{A}) \simeq \mathbf{Z}_2$. Using a classic theorem of isomorphism for the groups, we obtain that

$$C_n(p,q) \simeq \frac{RO(p+1,q+1)}{\mathcal{A}} \simeq \frac{\psi(RO(p+1,q+1))}{\psi(\mathcal{A})} \simeq \frac{O(p+1,q+1)}{\mathbf{Z_2}} \simeq PO(p+1,q+1).$$

Such a result is in agreement with the construction made by R. Penrose of the classical conformal group.[37]

2.4.2.4.3 Corollary

$$\tilde{\varphi}(RO^+(p+1,q+1)) \simeq \frac{RO^+(p+1,q+1)}{\mathcal{A}} \simeq \frac{\psi(RO^+(p+1,q+1))}{\psi(\mathcal{A})} \simeq \frac{O^+(p+1,q+1)}{\mathbf{Z}_2} \simeq PSO(p+1,q+1).$$

[36] Cf., for example, A. Crumeyrolle, Structures spinorielles, *Annales de l'I.H.P., section A*, vol. XI, no. 1, 1969, pp. 19–55.

[37] Cf., for example, R. Penrose, (a) Twistor algebra, *J. of Math. Physics*, t. 8, 1967, pp. 345–366; (b) Twistor quantization and curved space time, *Int. J. of Th. Physics*, (I), 1968. R. Penrose, for $p = 1, q = 3$, defines the conformal group as a group of transformations of $\frac{S^1 \times S^3}{\mathbf{Z}_2}$.

$g \in \mathrm{Pin}(p+1, q+1)$	$f \in C_n(p,q)$; $f = \tilde{\varphi}(g)$	$\sigma_g(x)$
$g = \exp b = \exp\left(\frac{1}{2} b^{ji} e_{ij}\right)$ $b^{ij} = \frac{1}{2} h^{ij} : hx = bx = xb$	"proper rotation" $\in SO_+(p,q)$: $x \to t(x) = g(x)g^{-1}$; $t = \exp h, t = \tilde{\varphi}(g)$	1
$g = e_{ij}$ $1 \le i < j \le n$	$t \in SO(p,q) : x \to t(x) = u_{(i)} \circ u_{(j)}(x)$; $u[u_{(i)} \circ u_{(j)}(x)] = u_{(i)} \circ u_{(j)} \circ u(x)$	1
$g = e_i$ $1 \le i \le n$	$t \in O(p,q) - SO(p,q)$: $x \to t(x) = u_{(i)}(x)$; $u(u_{(i)}(x)) = u_{(i)}(u(x))$	1
$g = \exp\left(\frac{1}{2}(e_{n+1} + e_0)\right) y$ $= 1 + \frac{1}{2}(e_{n+1} + e_0)y$; $y \in E_n$	$x \to x + y$ translation of E_n	1
$g = \exp\left(\frac{1}{2}\eta e_0 e_{n+1}\right)$, $\eta \in \mathbf{R}^*$	$x \to \lambda x$ dilation of E_n, $\lambda = \exp(-\eta)$, $\lambda \in \mathbf{R}^{*+}$	λ^{-1}
$g = e_0$ $g = e_{n+1}$	$x \to \frac{1}{x}$: inversion of center 0 and power 1: $x^2 u(x^{-1}) = u_{(0)}(u(x))$, for $x \neq 0$ $x \to \frac{-1}{x}$: inversion of center 0 and power -1, $(-x^2)u(-x^{-1}) = u_{(n+1)}(u(x))$, for $x \neq 0$	x^2 $-x^2$
$g = \exp\left(\frac{1}{2}(e_{n+1} - e_0)a\right)$ $= 1 + \frac{1}{2}(e_{n+1} - e_0)a$, $a \in E_n$	$x \to x(1+ax)^{-1}$ transversion of E_n, for $x : (1+ax)^2 \neq 0$	$N(1+ax)$

Table of results of 2.4.2.3

Thus $\tilde{\varphi}(RO^+(p+1, q+1)) = (C_n(p,q))_r$ is isomorphic to the special projective group $PSO(p+1, q+1)$.

2.4.2.5 Connected Components of $C_n(p,q)$, $n > 2$

We denote by $\boxed{G}$ the connected component of identity for the topological group G. Classically,

$$\frac{C_n(p,q)}{\boxed{C_n(p,q)}} \simeq \frac{RO(p+1,q+1)}{\tilde{\varphi}^{-1}\left(\boxed{C_n(p,q)}\right)},$$

according to a well-known result of algebra. Since $\boxed{C_n(p,q)}$ is connected, if an element of $RO(p+1, q+1)$ is mapped by $\tilde{\varphi}$ into $\boxed{C_n(p,q)}$, the same is true for any element in its connected component. Conversely, any element in $\boxed{C_n(p,q)}$ has its preimage in one of the connected components of the elements of Ker $\tilde{\varphi}$, as it is shown by the permissible pullback of a path of $\boxed{C_n(p,q)}$ into $RO(p+1, q+1)$, since $RO(p+1, q+1)$ is a covering group of $C_n(p,q)$. Therefore, the preimage by $\tilde{\varphi}$ of $\boxed{C_n(p,q)}$ is the set of the connected components of the elements of Ker $\tilde{\varphi}$ in $RO(p+1, q+1)$. We use previous notations of 1.2.2.6 and 1.2.2.7. $RO^+(p+1, q+1)$ denotes $RO(p+1, q+1) \bigcap C^+(E)$.

Let G_0 denote the subgroup of elements g of $RO(p+1, q+1)$ with $N(g) = 1$ and let $G_0^+ = G_0 \cap RO^+(p+1, q+1)$. $G_0^+ = \mathrm{Spin}(p+1, q+1)$ with our notation (cf. Chapter 1). It is well known[38] that G_0^+ is connected and of index 2 in $RO^+(p+1, q+1)$. But $RO(p+1, q+1) = RO^+(p+1, q+1) \cup C$, where $C_C = C_{RO(p+1,q+1)} RO^+(p+1, q+1)$ with $RO^+(p+1, q+1) \cap C = \emptyset$. C possesses two connected components since the product of an element in $RO^+(p+1, q+1)$ by a nonisotropic vector z with $N(z) = 1$ is a bijective mapping from $RO^+(p+1, q+1)$ onto C.

$RO(p+1, q+1)$ possesses four connected components: G_0^+, $G_0 - G_0^+$, $C_C G_0$ the complement in C of G_0, $RO^+(p+1, q+1) - G_0^+$. ± 1 *belong to* G_0^+. An easy computation shows that $N(e_N) = (-1)^{q+1}$.

(a) *$p + q = n$ even*

(α) *pq even* (p and q even)
$N(e_N) = -1, \pm e_N \in RO^+(p+1, q+1) - G_0^+, \tilde{\varphi}^{-1}\left(\boxed{C_n(p,q)}\right) = RO^+(p+1, q+1)$. It is well known that $RO(p+1, q+1)/RO^+(p+1, q+1) \simeq \mathbf{Z}_2$.[39] *$C_n(p,q)$ has two connected components.*

(β) *pq odd* (p and q odd)
$N(e_N) = 1, \pm e_N \in G_0^+, \tilde{\varphi}^{-1}\left(\boxed{C_n(p,q)}\right) = G_0^+$, $RO^+(p+1, q+1)/G_0^+(p+1, q+1) \simeq \mathbf{Z}_2$.[40] *$C_n(p,q)$ has four connected components.*

(b) *$p + q = n$ odd*

(α) *p even, q odd*
$N(e_N) = 1, \pm e_N \in G_0 \setminus G_0^+$. *$C_n(p,q)$ possesses two connected components.*

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[38] Cf., for example, C. Chevalley, *The Algebraic Theory of Spinors*, op. cit.; A. Crumeyrolle, Structures spinorielles, *Annales de l'I.H.P., section A*, vol. VI, no. 1, 1969, pp. 19–55; R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit.

[39] Idem.

[40] Idem.

(β) *p odd, q even*
$N(e_N) = -1, \pm e_N \in C_C G_0$. *$C_n(p, q)$ has two connected components.*

We have obtained the following statement:

2.4.2.5.1 Proposition *If pq is odd, $C_n(p, q)$ has four connected components, and if pq is even, $C_n(p, q)$ has two connected components.*

2.4.2.6 Consequences

2.4.2.6.1 Connected Component of the Identity in $C_n\ (p,\ q)$

Classically, in the Lie group $RO(p + 1, q + 1)$ one can find a neighbourhood of the identity element generated by the exponential mapping.[41] According to the table given in 2.4.2.4, we can deduce that the generic element of the connected component of the identity element in $C_n(p, q)$ is a composite of images by $\tilde{\varphi}$ of elements in $RO(p + 1, q + 1)$ such as $\exp\left(-\frac{1}{2}b^{ji}e_{ij}\right)$, $\exp\left(\frac{1}{2}(e_{n+1} + e_0)y\right)$, $y \in E_n$, $\exp\left(\frac{1}{2}\eta e_0 e_{n+1}\right)$, with $\eta \in \mathbf{R}^*$, $\exp\left(\frac{1}{2}(e_{n+1} - e_0)a\right)$, $a \in E_n$. We will use this remark later.

2.4.2.6.2 Topological Remarks

a) It is known[42] that $O(p) \times O(q)$ is a maximal compact subgroup of $O(p, q)$ and that every compact subgroup of $O(p, q)$ is conjugate to a subgroup of $O(p) \times O(q)$. More precisely, $O(p, q)$ is *homeomorphic* to $O(p) \times O(q) \times \mathbf{R}^{pq}$.[43] Thus, since the Poincaré group $P(p, q)$ is the semidirect product of the Lorentz group $O(p, q)$ and the group of translations of E_n that has n parameters, we have obtained that $P(p, q)$ is *homeomorphic* to $O(p) \times O(q) \times \mathbf{R}^{pq+n}$; since the conformal affine group is by definition the semidirect product of $P(p, q)$ and of the group of positive dilations, observing that $\mathbf{R}^+$ is homeomorphic to $\mathbf{R}$, we obtain that the conformal affine group is *homeomorphic to $O(p) \times O(q) \times \mathbf{R}^{pq+n+1}$*. (Cf. below footnote 120 in 2.9.1.3.3; see also: S. Kobayashi, Transformations groups in differential geometry, op.cit. p. 10. The conformal affine group is the semidirect product of $CO(p, q)$, defined in Kobayashi, p. 10, and of the translation group of $E_n(p, q)$.)

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[41] Cf., for example, C. Chevalley, *Theory of Lie Group, I*, Princeton University Press, Princeton, 1946 (fifth printing, 1962).

[42] J. A. Wolf, *Spaces of Constant Curvature*, Third edition, Publish or Perish, Inc., Boston, Mass., 1974.

[43] Such a result is in agreement with a general theorem of E. Cartan, improved by K. Iwasawa according to which any Lie group is homeomorphic to the topological product of a compact Lie group and a vector space (cf. S. Helgason, *Differential Geometry and Symmetric Spaces*, op. cit., p. 240).

We will use in 2.5.1.6 the following result:
The $\boxed{C_n(p,q)}$ connected component of the identity in $C_n(p,q)$ is homeomorphic to $\boxed{\text{O(p,q)}} \times \mathbf{R}^{2n+1}$.

Such a result comes from the fact that according to 2.4.2.6.1 any element of the connected component of the whole conformal group $C_n(p,q)$ can be written as the product of a proper rotation belonging to $\boxed{\text{O(p,q)}}$, of a translation, of proper dilation and of special conformal transformation, taking account of the facts that the group of translations has n parameters, the Poincaré group $\frac{n(n+1)}{2}$ parameters, the conformal affine group $\frac{n(n+1)}{2}+1$ parameters, and the whole conformal group $C_n(p,q)$ has $\frac{(n+1)(n+2)}{2}$ parameters.

According the remarks given above in 2.4.2.6.2.a we obtain that $\boxed{C_n(p,q)}$ is homeomorphic to $SO^+(p,q) \times \mathbf{R}^{pq+2n+1}$ and homeomorphic to $SO(p) \times SO(q) \times \mathbf{R}^{pq+2n+1}$. Moreover, $C_n(p,q)$ isomorphic to $PO(p+1,q+1)$ is homeomorphic to $O(p+1) \times O(q+1) \times \mathbf{R}^{(p+1)(q+1)}/\mathbf{Z}_2$. We will use these remarks below in 2.5.1.6.

(b) It is easy to show that $\pm e_N = \pm(e_0 e_{n+1} e_1 \cdots e_n) \notin \text{Spin}(p,q) = G_0^+(p,q)$.[44] Since, classically, $\psi^{-1}\left(\boxed{O(p,q)}\right) = G_0^+(p,q)$,[45] the restriction of $\tilde{\varphi}$ to $G_0^+(p,q)$ is identical to the standard homomorphism ψ (often called twisted projection) and to the classical homomorphism φ.[46]

2.4.3 Covering groups of the complex conformal group C_n

2.4.3.1 Some classical reviews (cf. for example C. Chevalley, The algebraic theory of spinors, op.cit., pp. 40–41 and pp. 60–61)

Let $E_n(p,q)$ be $\mathbf{R}^{p+q}$, with $p+q=n>2$, endowed with a quadratic form Q of signature (p,q) and the corresponding bilinear form B. Let E'_n, denoted also by $(E_n)_{\mathbf{C}}$, be the complexification of E_n. E'_n is an n–dimensional $\mathbf{C}$–space. If $\{e_j\}_{1\le j\le n}$ denotes the standard orthonormal basis for E_n, then $\{1\otimes e_j\}_{1\le j\le n}$ is a basis for E'_n over the field $\mathbf{C}$. If F denotes any n–dimensional $\mathbf{C}$–space, the real space obtained

[44] In any case, n even or odd, any element in $RO(p,q)$ is classically a product of elements v_i in $E_n(p,q)$ with $N(v_i)=q(v_i)=\pm 1$. Since $e_1\cdots e_n$ is in $RO(p,q)$ and since e_0 and $e_{n+1} \notin E_n(p,q)$, $e_0 e_{n+1} \notin RO(p,q)$ but $e_0 e_{n+1} \in RO(1,1)$. Thus, $e_N \notin RO(p,q)$, since if not, $e_0 e_{n+1} = e_N (e_1 \cdots e_n)^{-1}$ would be in $RO(p,q)$, a contradictory result.

[45] Cf. Chapter 1 or A. Crumeyrolle, Structures spinorielles, *Ann. Inst. H. Poincaré, section A*, vol XI, no 1, 1969, pp. 19–55.

[46] For any x in $E_n(p,q)$ and any g in $G_0^+(p,q)$, let us recall that $\varphi(x).g = gxg^{-1}$ and $\psi(g).x = \pi(g).xg^{-1}$.

by restriction of the scalars to the field **R** is denoted by ${}_{\mathbf{R}}F$. If $\{e_j\}_{1\leq j\leq n}$ is a **C**–basis of F, then $\{e_j, \imath e_j\}_{1\leq j\leq n}$ is an **R**–basis of ${}_{\mathbf{R}}F$, that is a $2n$–dimensional **R**–space.

Let B' be the bilinear form obtained from B by extension of **R** to the overfield **C** and let Q' be the corresponding quadratic form obtained by extension of Q. It is shown in C. Chevalley (op.cit II,5 p. 41) that $C'_n(Q)$, the complexification of $C_n(Q)$, is isomorphic to the Clifford algebra of Q'.

2.4.3.2 Definition Let f be a continuously differentiable mapping from an open set U' of E'_n into E'_n. Then f is said to be conformal in U' if there exists a continuous function λ' from U' into $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$ such that for almost all $x \in U'$ and for all $a, b \in E'_n$ we have that $B'(d_x f(a), d_x f(b)) = \lambda'^2(x)\, B'(a, b)$. The set of such mappings constitutes a group denoted by $Conf'(n)$ or sometimes by C'_n, if there is not any ambiguity.

2.4.3.2 Covering group of C'_n

The previous route used in 2.4.2.2 can be taken again. Let $\{e_j\}_{1\leq j\leq n}$ be the standard basis of E'_n. Let F be $E'_n \oplus E'_2$, where E'_2 is a complex hyperbolic plane with its standard orthonormal basis $\{e_0, e_{n+1}\}$, $Q'(e_0) = 1$, $Q'(e_{n+1}) = -1$. We can construct an injective mapping u', from E'_n into the isotropic cone of $F' = E'_{n+2}$, defined by

$$u'(x) = \frac{1}{2}\left(Q'(x) - 1\right) e_0 + x + \frac{1}{2}\left(Q'(x) + 1\right) e_{n+1}, \text{ with } Q'(x) = x^2,$$

and a homomorphism $\tilde{\varphi}'$ with a discrete kernel $\mathcal{A}'$ from $RO'(n+2)$—with obvious notation the $'$ are relative to the complex case—onto $\tilde{\varphi}'(RO'(n+2)$ such that for almost all $x \in E'_n$ and for any $g \in RO'(n+2)$, if we set $\tilde{\varphi}'(g) = f$, there exists $\sigma_g(x) \in \mathbf{C}$ such that we have $\pi(g)u'(x)g^{-1} = \sigma_g(x)u'(\tilde{\varphi}'(g).x)$.

As stated previously $\tilde{\varphi}'(RO'(n=2))$ can be identified with the group generated by inversions, similarities and translations of E'_n : $\tilde{\varphi}'(RO'(n+2)) = Conf'(n)$. The group $\tilde{\varphi}'(RO'^+(n+2))$ is called the restricted complex conformal group and denoted by $Conf'(n)_r$. We can verify that

$$Conf'(n) \simeq \frac{RO'(n+2)}{\mathcal{A}'}$$

and respectively

$$Conf'(n)_r \simeq \frac{RO'^+(n+2)}{\mathcal{A}'},$$

where $\mathcal{A}' = \{1, -1, e_N, -e_N, i, -i, ie_N, -ie_N\}$, with $e_N = \{e_0 e_{n+1} e_1 \ldots e_n\}$. The construction of the corresponding table of results is left as an exercise.

2.5 Real Conformal Spinoriality Groups and Flat Real Conformal Geometry

2.5.1 Conformal Spinoriality Groups

2.5.1.1 A Brief Review of Previous Results

Let $E_n(p,q)$ $(p+q=n, n>2)$ be $\mathbf{R}^n$ with a quadratic form Q of arbitrary signature (p,q). $Cl(E_n)$ denotes the Clifford algebra of E_n with the quadratic form Q; π is the principal automorphism of $Cl(E_n)$, τ the principal antiautomorphism of $Cl(E_n)$. B is the fundamental bilinear form associated with Q chosen so that for all $x \in \mathbf{R}^n$, $B(x,x) = Q(x)$. We recall that the group $ROQ = RO(p,q)$ constitutes the 2-fold covering of the orthogonal group $O(p,q)$. If $g \in \text{Pin}(p,q)$, we define $\varphi(g)x = gxg^{-1}$, $x \in \mathbf{R}^n$, $\varphi(g) \in O(p,q)$ and $\psi(g) = \pi(g)xg^{-1}$, $\psi(g) \in O(p,q)$. We introduce an orthonormal basis of $E_n(p,q)$ such that $Q(e_i) = e_i^2 = \varepsilon_i$ ($\varepsilon_i = 1$, $1 \le i \le p$, $\varepsilon_i = -1$, $p+1 \le i \le n$). In $\mathbf{R}^2$ with a quadratic form Q_2 of signature $(1, 1)$, we consider an orthonormal basis $\{e_0, e_{n+1}\}$ such that $Q(e_0) = (e_0)^2 = 1$, $Q(e_{n+1}) = (e_{n+1}))^2 = -1$. Then $\{e_1, \ldots, e_n, e_0, e_{n+1}\}$ is an orthonormal basis of $\mathbf{R}^{n+2} = E_{n+2}(p+1, q+1) = E_n(p,q) \oplus E_2(1,1)$; e_0 and e_{n+1} are chosen once and for all. $C_n(p,q)$ stands for the conformal Lie group of $\mathbf{R}^n$ isomorphic to $PO(p+1, q+1) = \frac{O(p+1,q+1)}{\mathbf{Z}_2}$, which we agree to call the Möbius group of $E_n(p,q)$. More precisely, we have constructed an injective mapping u from $E_n(p,q)$ into the isotropic cone C_{n+2} of $E_{n+2}(p+1, q+1)$ defined for all $x \in E_n(p,q)$ by

$$u(x) = \frac{1}{2}(x^2 - 1)e_0 + x + \frac{1}{2}(x^2 + 1)e_{n+1}. \tag{B}$$

The "projection" $\tilde{\varphi}$ called "twistor projection" or "conformal spinor projection" from $RO(p+1, q+1)$ onto $C_n(p,q)$ is such that for almost all $x \in E_n(p,q)$ and for all $g \in RO(p+1, q+1)$,

$$\pi(g)u(x)g^{-1} = \psi(g)u(x) = \sigma_g(x)u(\varphi(g)x), \tag{A}$$

with $\sigma_g(x) \in \mathbf{R}$. We set $e_N = e_0 e_{n+1} e_1 \cdots e_n$: the kernel of $\tilde{\varphi}$: $\mathcal{A} = \{1, -1, e_N, -e_N\}$ isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$ if $(e_N)^2 = 1$, or to $\mathbf{Z}_4$ if $(e_N)^2 = -1$. $\tilde{\varphi}(RO^+(p+1, q+1))$, is called *the real conformal restricted group.*

If we set $x_0 = (e_0 + e_{n+1})/2$ and $y_0 = (e_0 - e_{n+1})/2$, $\{x_0, y_0\}$ is then a special "real Witt basis" of $\mathbf{C}^2$ associated with $\{e_0, e_{n+1}\}$. From (B): $u(x) = x^2 x_0 + x - y$, we deduce

$$u(x)y_0 = x^2 x_0 y_0 + x y_0 \quad \text{and} \quad y_0 u(x) = x^2 y_0 x_0 + y_0 x,$$

whence we obtain

$$u(x)y_0 + y_0 u(x) = 2B(u(x), y_0) = x^2.$$

Thus (A) is equivalent to

$$x = u(x) - 2B(u(x), y_0)x_0 + y_0, \tag{A$_1$}$$

subject to $(u(x))^2 = 0$, since $x_0 y_0 + y_0 x_0 = 2B(x_0, y_0) = 1$ and $x y_0 + y_0 x = 2B(x, y_0) = 0$.

2.5.1.2 Fundamental Diagram

It is now possible to construct an explicit homomorphism h from the orthogonal Lie group $O(p+1, q+1)$ onto $C_n(p, q)$, in order to obtain a commutative diagram.[47] i, respectively j, denotes the identity map from $O(p, q)$ into $C_n(p, q)$, respectively $O(p+1, q+1)$.

First, we construct h. For any ω belonging to $O(p+1, q+1)$, there exists g(modulo ± 1) belonging to $RO(p+1, q+1)$ such that $\omega = \psi(g)$. Since according to [1] for any g, g' in $RO(p+1, q+1)$ such that $\tilde{\varphi}(g) = f \in C_n(p, q)$ and $\tilde{\varphi}(g') = f' \in C_n(p, q)$, $\tilde{\varphi}(g'g) = f' \circ f$ and for almost all $x \in E_n(p, q)$, $\sigma_{g'g}(x) = \sigma_{g'}(f(x))\sigma_g(x)$, we obtain that $\sigma_g(x) \neq 0$ when $f(x)$ is defined and that (A) is equivalent to

$$u(f(x)) = \lambda_g(x)\psi(g)u(x) \quad \text{for } f = \tilde{\varphi}(g), \tag{A$_2$}$$

where $\lambda_g(x) = (\sigma_g(x))^{-1}$. So with any $\omega \in O(p+1, q+1)$ we can associate $f = \tilde{\varphi}(g) \in C_n(p, q)$ such that $f(x) = \lambda_g(x)\{\omega.u(x) - 2B(\omega.u(x), y_0)x_0\} + y_0$ with $2\lambda_g(x)B(\omega.u(x), x_0) = -1$. One obtains a map h from $O(p+1, q+1)$ into $C_n(p, q)$.

We agree to set $\lambda_g = \lambda_{-g} = \lambda_\omega$, where $\omega = \psi(g) = \psi(-g) \in O(p+1, q+1)$, and *we can easily verify that it is possible to write*

$$f(x) = \lambda_\omega(x)\{\omega.u(x) - 2B(\omega.u(x), y_0)x_0\} + y_0, \quad \lambda_\omega(x) = \frac{-1}{2B(\omega.u(x), x_0)} \tag{C}$$

when $f(x)$ is defined and that the diagram is commutative.

2.5.1.2.1 Proposition *One can verify that $\omega \to h(\omega) = f = \tilde{\varphi}(g)$ is a homomorphism*[48] *from $O(p+1, q+1)$ onto $C_n(p, q)$ such that $i = h \circ j$, $\tilde{\varphi} = h \circ \psi$. Thus we obtain an isomorphism h_1 of Lie groups from $PO(p+1, q+1)$ onto $C_n(p, q)$ by using quotient groups such that $h = h_1 \circ \tilde{h}$, where $\tilde{h}$ is the homomorphism associated with the classical exact sequence of groups*

$$1 \to \mathbf{Z}_2 \to O(p+1, q+1) \to PO(p+1, q+1) \to 1.$$

Let k_1 be the inverse of h_1. In the same way as previously, if C'_n stands for the complex conformal group[49] and $O'(n+2)$ for the complex orthogonal group,

47 P. Anglès, (a) Géométrie spinorielle conforme orthogonale triviale et groupes de spinorialité conformes, *Report HTKK Mat A* 195, pp. 1–36, Helsinki University of Technology, 1982. (b) Real conformal spin structures on manifolds, *Scientiarum Mathematicarum Hungarica*, vol. 23, pp. 115–139, Budapest, Hongary, 1988.

48 Cf. exercises below.

49 Cf. above 2.4.3.

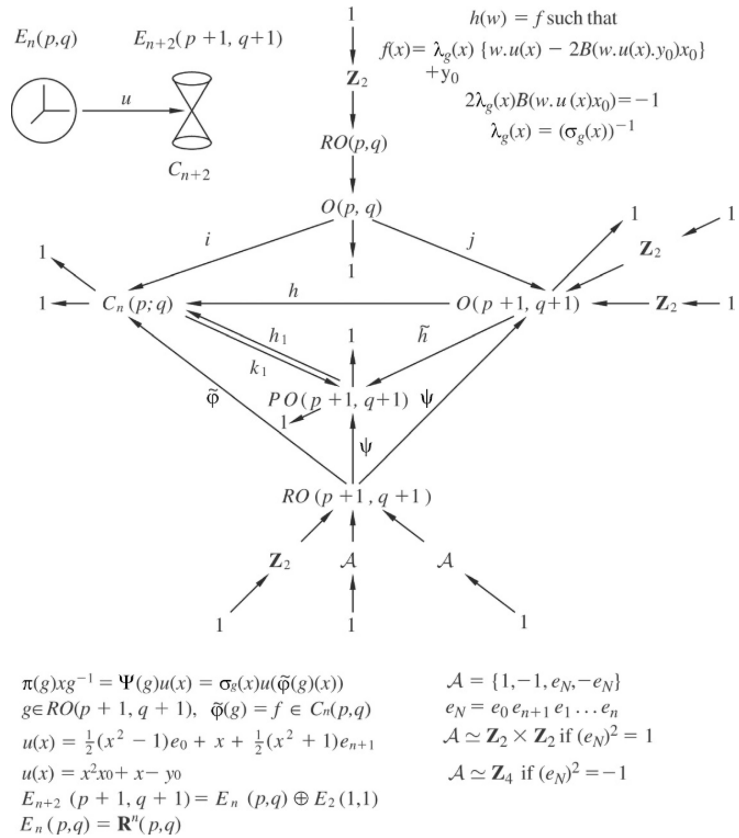

Fig. 2.1. Fundamental diagram

$RO'(n+2)$ is an 8-fold covering of C'_n, with kernel

$$\mathcal{A}' = \{1, -1, e_N, -e_N, i, -i, ie_N, -ie_N\}.\tilde{\varphi}',$$

respectively ψ', denotes the "complex conformal spinor projection," respectively the "twisted spinor projection," from $RO'(n+2)$ onto C'_n, respectively $O'(n+2)$. Thus, one can construct a diagram analogous to the fundamental diagram in Fig. 2.1—this is left as an exercise.

2.5.1.2.2 Remark

Let us, finally recall[50] the following remark: if $n = 2r$, then $e_N f_{r+1} = (-i)^{r-p} f_{r+1}$, where $f_{r+1} = y_1 \cdots y_r y_0$ is an $(r+1)$-isotropic vector and $f_{r+1}e_N = (-1)^{r+1} \cdot (-i)^{r-p} f_{r+1}$ according to a result given in C. Chevalley[51] (cf. exercises below).

[50] P. Anglès, Les structures spinorielles conformes réelles, Thesis, op. cit., p. 41.

[51] C. Chevalley, *The Algebraic Theory of Spinors* op. cit., p. 91.

2.5.1.3 Definitions of Real Conformal Spinoriality Groups (n Even, $n = 2r, r > 1$)

Let $f_{r+1} = y_1 \cdots y_r y_0 = f_r y_0$ be an $(r + l)$-isotropic vector.

2.5.1.3.1 Definition Let H_C be the set of elements $\gamma \in RO^+(p+1, q+1)$ such that $\gamma f_{r+1} = \varepsilon_1 f_{r+1}$, where $\varepsilon_1 \in \mathcal{A} = \{1, -1, e_N, -e_N\}$. We agree to call, by definition, the subgroup $S_C = \tilde{\varphi}(H_C)$ of $(C_n(p, q))_r$ the real conformal group associated with $f_{r+1} = y_1 \cdots y_r y_0$.

Following the result given above (according to which $e_N f_{r+1} = (-i)^{r-p} f_{r+1}$), such a definition is equivalent to the following one:

$S_C = \tilde{\varphi}(H_C)$, where H_C is the set of elements $\gamma \in RO^+(p + 1, q + 1)$ such that if $r - p \equiv 0$ or 2 (modulo 4), $\gamma f_{r+1} = \pm f_{r+1}$, and if $r - p \equiv 1$ or 3 (modulo 4), $\gamma f_{r+1} = \varepsilon f_{r+1}$ with $\varepsilon = \pm 1$ or $\pm i$.

2.5.1.3.2 Definition Let $(H_C)_e$ be the set of elements $g \in RO^+(p + 1, q + 1)$ such that $g f_{r+1} = \mu f_{r+1}$, where $\mu \in \mathbf{C}^*$. We agree to call by definition the enlarged real conformal spinoriality group associated with f_{r+1} the subgroup $(S_C)_e = \tilde{\varphi}((H_C)_e)$ of $(C_n(p, q))_r$.

We can observe that e_0 and e_{n+1} being chosen once and for all, these definitions are associated with the choice of an r-isotropic vector $f_r = y_1 \cdots y_r$ of $E_n(p, q)$.

2.5.1.3.3 Remark Let us observe that these subgroups, at first glance "bigger" than those defined by Crumeyrolle[52] are subgroups of $(C_n(p, q))_r$ that cannot be reduced to subgroups of $SO(p, q)$ defined as real spinoriality groups.[53] More precisely, one can easily verify, for example, that any real conformal spinoriality group contains the following elements:

(α) the special conformal transformation $x \to f(x) = x(1 + ax)^{-1}$, where, $f = \tilde{\varphi}\big(1 + \frac{1}{2}(e_{n+1} - e_0)a\big)$ with $a = e_1 + \cdots + e_n$,

(β) the translation $x \to x + y$, where $y = e_1 + \cdots + e_p - e_{n-p+1} \cdots - e_n$.

(γ) We notice that $\tilde{\varphi}(e_0 e_{n+1}) = -Id_{E_n}$ belongs to S_C and that the following elements of $SO(p, q)$: $\tilde{\varphi}(e_i e_{n-i+1})$ belong to S_C for all i, $1 \le i \le n$.

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[52] A. Crumeyrolle, (a) Groupes de spinorialité, *Annales de l'Inst. H. Poincaré Section A (N.S.)*, 14, 1971, pp. 309–323; (b) Dérivations, formes et opérateurs usuels sur les champs spinoriels des variétés différentiables de dimension paire, ibid. 16, 1972, pp. 171–201; (c) Fibrations spinorielles et twisteurs généralisés, *Period. Math. Hungar.*, 6, 1975, pp. 143–171; (d) Spin fibrations over manifolds and generalized twistors, Differential geometry, *Proc. Sympos. Pure Math.*, Vol. 27, Part 1, Stanford Univ., Stanford, Calif., 1973, Amer. Math. Soc., Providence, R.I., 1975, pp. 53–67.

[53] Idem.

2.5.1.4 Description of the Enlarged Real Conformal Spinoriality Groups

The abbreviation m.t.i.s. stands for maximal totally isotropic subspace as in C. Chevalley.[54]

2.5.1.4.1 Proposition *Any enlarged real conformal group of spinoriality* $(S_C)_e$ *is the stabilizer of the m.t.i.s. associated with the r-isotropic vector* $y_1 \cdots y_r$ *for the action of* $(C_n(p,q))_r$. *If pq is even,* $(S_C)_e$ *is connected; if pq is odd,* $(S_C)_e$ *has two connected components. For* $0 < p < r$, $\dim(S_C)_e = (r+1)^2 + p(p+1)/2$. $(\sigma)_e$, *the enlarged real group of spinoriality associated with* f_r *as by A. Crumeyrolle*[55] *is a normal subgroup of* $(S_C)_e$.

Proof. The demonstration can be carried out in two steps. Let us write $f_1 = h(\omega)$ for $\omega \in O(p+1, q+1)$.

(a) *First, we suppose that* $u(f_1(y_i)) = f_1(y_i) - y_0$ *is well determined for all i,* $1 \leq i \leq r$, *that is, equivalently,* $f_1(0)$ *and* $f_1(y_i)$ *well defined for all i,* $1 \leq i \leq r$.

According to a result by Crumeyrolle[56], $\gamma f_{r+1} = \pm\mu f_{r+1}$, $\mu \in \mathbf{C}^*$, is equivalent to $\gamma f_{r+1}\gamma^{-l} = N(\gamma)\mu^2 f_{r+1}$. Thus, $(H_C)_e$ is the set of elements $g \in RO^+(p+1, q+1)$ such that $g f_{r+1} g^{-1} = \sigma f_{r+1}$, where $\sigma = N(g)\mu^2 = \pm\mu^2$.

One can easily notice that $\pi(g) f_{r+1} g^{-1} = \sigma f_{r+1}$ is equivalent to

$$\pi(g) y_1 g^{-1} \pi(g) y_2 g^{-1} \cdots \pi(g) y_r g^{-1} \pi(g) y_0 g^{-1} = \sigma f_{r+1}. \qquad \text{(I)}$$

We set $\psi(g) = \omega$, $\omega = SO(p+1, q+1)$, so that $\pi(g) y_i g^{-1} = \omega(y_i)$, $1 \leq i \leq r$, and $\pi(g) y_0 g^{-1} = \omega(y_0)$. So g belongs to $(H_C)_e$ iff $\psi(g) = \omega$ belongs to σ_e, *the real enlarged spinoriality group associated with* $f_{r+1} = y_1 \cdots y_r y_0$.

According to the diagram given above, we obtain $(S_C)_e = h(\sigma_e)$.

By an easy computation, taking account of the formulas (C), we obtain that (I) is equivalent to

$$\begin{aligned}&\sigma_\omega(y_1)\sigma_\omega(y_2)\cdots\sigma_\omega(y_r)(-\sigma_\omega(0))u(f_1(y_1))\cdots u(f_1(y_r))u(f_1(y_0))\\ &\quad= \sigma y_1 \cdots y_r y_0 = -\sigma u(y_1)\cdots u(y_r)u(0),\end{aligned}$$

Since $u(y_1)\cdots u(y_r)u(0) = -y_1 \cdots y_r y_0$, we notice that the m.t.i.s. associated respectively with $u(y_1)\cdots u(y_r)u(0)$ and with $y_1 \cdots y_r y_0$ are equal. So we have the following relation equivalent to (I):

$$\sigma_\omega(y_1)\cdots\sigma_\omega(y_r)\sigma_\omega(0))u(f_1(y_1))\cdots u(f_1(y_r))u(f_1(0)) = \sigma u y_1 \cdots u y_r u(0), \quad \text{(II)}$$

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[54] C. Chevalley, *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954.

[55] A. Crumeyrolle, *Fibrations Spinorielles et Twisteurs Généralisés*, op. cit.

[56] A. Crumeyrolle, *Fibrations Spinorielles et Twisteurs Généralisés*, op. cit.

which means[57] that the vectors $u(f_1(y_1)), \ldots, u(f_1(y_0)), u(f_1(0))$ belong to the $(r+1)$-m.t.i.s. F'_{r+1} associated with f_{r+1}.

We notice that u operates on the set of isotropic subspaces as the translation of vector $u(0) = -y_0$, since for any $z \in E(p,q)$, $u(z) = z^2 x_0 + z - y_0$. If $u(z)$ belongs to the $(r+1)$-m.t.i.s. F'_{r+1}, then z belongs to $F'_r = \{y_1, \ldots, y_r\}$.

According to our assumption, $\sigma_\omega(y_i) \neq 0$ for all i, $1 \leq i \leq r$, and $\sigma_\omega(0) \neq 0$. Since, taking into account another result of Crumeyrolle[58] ω belongs to σ_e, which stabilizes the $(r+1)$-m.t.i.s. $\{y_1, \ldots, y_r, y_0\}$ for the action of $SO(p+1, q+1)$, the restriction of ω to $F'_r = \{y_1, \ldots, y_r\}$ stabilizes F'_r. So we find that for all i, $1 \leq i \leq r$, $\sigma_\omega(y_i) = \sigma_\omega(0) \neq 0$.[59]

(I) is equivalent to

$$\omega(y_1)\cdots\omega(y_r)(-\sigma(0))f_1(0) + \omega(y_1)\cdots\omega(y_r)\sigma_\omega(0)y_0 = \sigma y_1 \cdots y_r y_0, \qquad \text{(III)}$$

since $\omega(y_0) = -\sigma_\omega(0)(f_1(0) - y_0)$. Since $\omega(y_1), \ldots, \omega(y_r)$ are independent in F'_r, according to the definition of ω, and since $f_1(0)$ belongs to F'_r,

$$\omega(y_1)\cdots\omega(y_r)f_1(0) = 0 \qquad \text{(III')}$$

necessarily.[60] Thus, (III) means that

$$\omega(y_1)\cdots\omega(y_r)y_0 = \frac{\sigma}{\sigma_\omega(0)} y_1 \cdots y_r y_0,$$

whence we deduce[61] that

$$\omega(y_1)\cdots\omega(y_r) = \frac{\sigma}{\sigma_\omega(0)} y_1 \cdots y_r. \qquad \text{(IV)}$$

An easy computation gives the following result:

$$\omega(y_1)\cdots\omega(y_r) = (\sigma_\omega(0))^r \prod_{i=1}^{r}(f_1(y_i) - f_1(0)), \qquad \text{(V)}$$

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[57] C. Chevalley, *The Algebraic Theory of Spinors*, op. cit.; S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964.

[58] A. Crumeyrolle, *Fibrations Spinorielles et Twisteurs Généralisés*, op. cit.

[59] According to the formulas (C) of 2.5.1.2, $\sigma_w(0) = -2B(w(-y_0), x_0) = 2B(w(y_0), x_0)$ and $\sigma_w(y_i) = -2B(w, u(y_i), x_0) = -2B(w(y_i - y_0), x_0) = 2B(w(y_0), x_0)$, whence the result.

[60] S. Sternberg, *Lectures on Differential Geometry*, op. cit., p. 14 Th. 4.3.2 chapter II.

[61] C. Chevalley, The algebraic theory of spinors, chapter III, p. 72, II.1.4. According to this result, since $y_0^2 = 0$, and since $A = \left(\omega(y_1)\ldots\omega(y_r) - \frac{\sigma'}{\sigma_\omega(0)} y_1 \ldots y_r\right) y_0$ is equal to zero, there exists a scalar μ such that $A = \mu y_0$ and because of the difference of degrees, necessarily $\mu = 0$. For additional information see also P. Anglès, Géométrie spinorielle conforme orthogonale triviale et groupes de spinorialité conformes, in the bibliography to this chapter.

as $\omega(y_i) = \sigma_\omega(0)(f_1(y_i) - f_1(0))$ for all i, $1 \leq i \leq r$.

- If $f_1(0) = 0$, we obtain that

$$f_1(y_1)\cdots f_1(y_r) = \frac{\sigma}{(\sigma_\omega(0))^{r+1}} y_1 \cdots y_r,$$

where $\sigma = \pm\mu^2$ belongs to $\mathbf{C}^*$. So μ_1 such that $f_1(y_1)\cdots f_1(y_r) = \mu_1 y_1 \cdots y_r$ is any element of $\mathbf{C}^*$.

- If $f_1(0) \neq 0$, observing that

$$\prod_{i=1}^{r}(f_1(y_i) - f_1(0)) f_1(0) = \frac{1}{(\sigma_\omega(0))^r}\omega(y_1)\cdots\omega(y_r) f_1(0) = 0,$$

according to (III′) and moreover noticing that the product on the left equals $f_1(y_1)\cdots f_1(y_r) f_1(0)$, we find that the vectors $f_1(y_1), \ldots, f_1(y_r), f_1(0)$ are dependent in F'_{r+1}.[62] [see IV and V]

$$\prod_{i=1}^{r}(f_1(y_i) - f_1(0)) = \frac{\sigma}{(\sigma_\omega(0))^{r+1}} y_1 \cdots y_r = \mu_1 y_1 \cdots y_r,$$

where μ_1 is any element in $\mathbf{C}^*$, taking account of the dependence of the vectors $f_1(y_1), \ldots, f_1(y_r), f_1(0)$, we obtain that $f_1(y_1)\cdots f_1(y_r) = \mu_2 y_1 \cdots y_r$, where $\mu_2 \in \mathbf{C}^*$.

(b) Let us prove now that for any f_1 belonging to $(S_C)_e$, it is permissible to suppose that $f_1(0)$ is well-defined and to find $z_1, \ldots, z_r$ linearly independent, belonging to F'_r such that all the $f_1(z_i)$ are well-defined if some of the elements $f_1(y_1), \ldots, f_1(y_r)$, are not defined. Let us recall that classically, for $x = x^1 e_1 + \cdots + x^p e_p + x^{p+1} e_{p+1} + \cdots + x^n e_n$ the x^i, $1 \leq i \leq p$, are called "spatial coordinates of $x \in E_n(p, q)$" and those for $p + 1 \leq i \leq n$ the "temporal coordinates of x." So, for any j, $1 \leq j \leq n$, we call $(\mathrm{Sym})_j$ the mapping already called u_{e_j} or u_j in 2.4.2.3. Moreover, $(\mathrm{Sym})_s$ denotes the space symmetry defined as the product $\prod_{1\leq j\leq p}(\mathrm{Sym})_j$, $(\mathrm{Sym})_t$ denotes the product $\prod_{p+1\leq j\leq p+q}(\mathrm{Sym})_j$, and $(\mathrm{Sym})_{st} = (\mathrm{Sym})_s(\mathrm{Sym})_t$. It is well known (see, for example *Spaces of Constant Curvature* by J. A. Wolf[63]) that

$$\frac{O(p+1, q+1)}{\boxed{(\mathrm{O(p+1,q+1)})}} \simeq \mathbf{Z}_2 \times \mathbf{Z}_2,$$

where $\boxed{G}$ denotes the connected component of the Lie group G, and that

$$\boxed{(O(p+1, q+1))} = SO^+(p+1, q+1).$$

[62] S. Sternberg, *Lectures on Differential Geometry*, Th. 4.2, p. 15.

[63] J. A. Wolf, *Spaces of Constant Curvature*, op. cit., p.341.

(i) *Let us assume that p and q are even ($n = p+q$ is even).* We know (2.4.2.6) that $\boxed{RO(p+1, q+1)} = G_0^+(p+1, q+1), \tilde{\varphi}^{-1}\left(\boxed{C_n(p,q)}\right) = RO^+(p+1, q+1)$, $C_n(p,q)$ has two connected components, and, finally, that $RO(p+1, q+1)$ has four connected components. We introduce the elements $\pm e_p$, $\pm e_Q$, where $e_p = e_0 e_1 \cdots e_p$ and $e_Q = e_{p+1} \cdots e_n e_{n+1}$. One may verify that ± 1 belong to $G_0^+(p+1, q+1)$, $\pm e_p$ belong to $G_0(p+1, q+1) - G_0^+(p+1, q+1)$, $\pm e_Q$ belong to $\mathcal{C}_C G_0(p+1, q+1)$, where C is $RO(p+1, q+1)\backslash RO^+(p+1, q+1)$. So, we find again that $O(p+1, q+1)$ has 4 connected components and that any element $\omega \in O(p+1, q+1)$ can be written $\omega = \omega^* \omega_0$, $\omega_0 \in \boxed{O(p+1, q+1)}$ and $\omega^* = \mathrm{Id}_{E_{n+2}} = \psi(\pm 1)$ or $\omega^* = \psi(\pm e_p) = (\mathrm{Sym})_s$ (space-symmetry), or $\omega^* = \psi(\pm e_Q) = (\mathrm{Sym})_t$ (time-symmetry), or $\omega^* = -\mathrm{Id}_{E_{n+2}} = \psi(\pm e_N) = (\mathrm{Sym})_{st}$ (space-time symmetry).

Such a result is well known and used in physics for $n = 4$, $p = 3$, and $q = 1$. Moreover we observe that e_P and e_Q belong to the same class of $RO(p+1, q+1)$ modulo $\mathcal{A} = \{1, -1, e_N, -e_N\}$, since $e_P(e_Q)^{-1} = \pm e_N$, as can be easily verified. Therefore necessarily,

$$\tilde{\varphi}(\pm e_P) = \tilde{\varphi}(\pm e_Q) = Inv(0,1) \circ (Sym)_s = (Sym)_t \circ Inv(0,-1)$$

in the space $E_n(p,q)$, where $Inv(0,1)$ (resp. $Inv(0,-1)$) denotes the inversion of center 0 and power 1 (resp. -1). We observe that $h\left(\boxed{O(p+1, q+1)}\right) \subset \boxed{C_n(p,q)}$ and that any element $f \in C_n(p,q)$ can be written $f = f^* \circ f_0$, where $f^* = h(\omega^*)$ and $f_0 = h(\omega_0) \in \boxed{C_n(p,q)}$ and $f^* = \tilde{\varphi}(\pm 1) = \varphi(\pm e_N) = (\mathrm{Id})_{E_n}$ or $f^* = \tilde{\varphi}(\pm e_p) = \tilde{\varphi}(\pm e_Q) = \mathrm{Inv}(0,1) \circ (\mathrm{Sym})_{\mathrm{e}} = (\mathrm{Sym})_t \circ \mathrm{Inv}(0,-1)$ in the space $E_n(p,q)$, where $\mathrm{Inv}(0,1)$, respectively $\mathrm{Inv}(0,-1)$, denotes the inversion of pole O and power 1, respectively of pole O and power -1. Therefore two cases appear: $f = f^* \circ f_0$, with f_0 belonging to $\boxed{C_n(p,q)}$, and $f^* = Id_{E_n}$ or $f^* = \mathrm{Inv}(0,1) \circ (\mathrm{Sym})_s = (\mathrm{Sym})_t \circ \mathrm{Inv}(0,-1)$. We recall that n and pq are even, and that $C_n(p,q)$ has two connected components. According to [2.4.2.6] $f_0 = \Omega \circ \mathcal{T} \circ \mathcal{H} \circ S$, where Ω belongs to $SO_+(p,q)$, $\mathcal{T}$ is a translation of vector $b \in E_n(p,q)$, $\mathcal{H}$ is a dilation, and S is the special conformal transformation or transversion $x \to x(1+ax)^{-1}$, where $a \in E_n(p,q)$. We remark that $(S_C)_e \subset \boxed{C_n(p,q)}$ and that we are led to study the case that f_0 belongs to $\boxed{C_n(p,q)}$, $f_0 = \Omega \circ \mathcal{T} \circ \mathcal{H} \circ S$, with

$$\sigma_{g_0}(x) = \sigma_\Omega \underbrace{(\mathcal{T} \circ \mathcal{H} \circ S(x))}_{=1} \underbrace{\sigma_{\mathcal{T}}(\mathcal{H} \circ S(x))}_{=1} \underbrace{\sigma_{\mathcal{H}} S(x)}_{=\lambda^{-1}} \underbrace{\sigma_S(x)}_{N(1+ax)} .$$

Since $\sigma_{g_0}(x) = \sigma_{\omega_0}(x) = 0$ is the equation of singular points of f_0,[64] we find that for any isotropic vector x, a singular point of f_0, we have $B(a,x) = -\frac{1}{2}$ since $N(1+ax) = 1 + 2B(a,x) + a^2x^2$. Since $f_0 = h(\omega_0)$ stabilizes[65] F'_r, *we observe*

[64] P. Anglès, Construction de revêtements du groupe conforme d'un espace vectoriel muni d'une métrique de type (p,q), *Ann Inst. H. Poincaré Sect. A (N.S.)*, 33 (1980), 33–51.

[65] A. Crumeyrolle, *Structures Spinorielles*, op. cit.

that $f_0(0)$ *is well-defined*, according to the fact that $f_0(0) = \Omega \circ \mathcal{T} \circ \mathcal{H} \circ S(0)$ with $S(0) = 0$, $\mathcal{H}(0) = 0$, and that $\Omega \circ \mathcal{T}(0)$ *belongs to* F'_r. Thus, we can assume that $f_1(0)$ is well-defined according to the writing of f_0. We have found that the condition $B(a, x) \neq \frac{1}{2}$ is sufficient for $f_0(x)$ to be well defined. This implies that x belongs to a dense subset of $E_n(p, q)$ (cf. also Jean Dieudonne in Elements d'Analyse, tome 3, Gauthier Villars, 1970, pp. 161–163 about Sard's theorem), in which we can choose suitable $z_1, \ldots, z_n$, such that all the $f_0(z_i)$ are well defined.

If y_i, $1 \leq i \leq r$, is a singular point for f_0, we can find an isotropic vector

$$a_1 = \sum_{i=1}^{r} \alpha^i y_i$$

belonging to F'_r such that $z_i = y_i + a_1$, $1 \leq i \leq r$, satisfy the conditions $B(a, z_i) \neq -\frac{1}{2}$, the vectors z_i being linearly independent. We note that setting $a_1 = y_1 + \cdots + y_r$, belonging to F'_r, $B(a, z_i) \neq -\frac{1}{2}$ for any i, $1 \leq i \leq r$, and these vectors z_i trans1ated from the y_i's are linearly independent.

(ii) *Let us now assume that p and q are odd.* $C_n(p, q)$ has 4 connected components and $\varphi^{-1}\left(\boxed{C_n(p, q)}\right) = G_0^+(p+1, q+1)$. We observe that $\pm 1, \pm e_N, \pm e_p, \pm e_Q$ belong to $G_0^+(p+1, q+1)$ and that $G_0^+(p+1, q+1) = \psi^{-1}\left(\boxed{O(p+1, q+1)}\right)$.[66] $RO(p+1, q+1)$ has four connected components as previously. There exists $e_0 e_{n+1} \in RO^+(p+1, q+1) \setminus G_0^+(p+1, q+1)$—even element with norm equal to -1; there exists $e_0 e_{n+1} e_1$—odd element with norm equal to -1, belonging to $C_C G_0(p+1, q+1)$; there exists $e_0 e_1 e_2$—odd element with norm equal 1, belonging to $G_0(p+1, q+1) \setminus G_0^+(p+1, q+1)$. Therefore any element ω of $O(p+1, q+1)$ can be written $\omega = \omega^* \omega_0$, where ω_0 belongs to $\boxed{O(p+1, q+1)}$ and

$$\omega^* = \mathrm{Id}_{E_{n+2}} = \psi(\pm 1) = \psi(\pm e_N) = \psi(\pm e_p) = \psi(\pm e_Q)$$

or

$$\omega^* = (\mathrm{Sym})_0 \circ (\mathrm{Sym})_{n+1} = \psi(e_0 e_{n+1})$$

or

$$\omega^* = (\mathrm{Sym})_0 \circ (\mathrm{Sym})_{n+1} \circ (\mathrm{Sym})_1 = \psi(e_0 e_{n+1} e_1)$$

or

$$\omega^* = (\mathrm{Sym})_0 \circ (\mathrm{Sym})_1 \circ (\mathrm{Sym})_2 = \psi(e_0 e_1 e_2).$$

Thus, any element f belonging to $C_n(p, q)$ can be written $f = f^* \circ f_0$, where $f_0 = h(\omega_0)$ belongs to $\boxed{C_n(p, q)}$ and where $f^* = \mathrm{Id}_{E_n}$ or $f^* = -\mathrm{Id}_{E_n}$ or $f^* = (-\mathrm{Id})_{E_n} \circ (\mathrm{Sym})_1$ or $f^* = \mathrm{Inv}(0, 1) \circ (\mathrm{Sym})_1 \circ (\mathrm{Sym})_2$ with obvious notations.

[66] A. Crumeyrolle, *Structures Spinorielles*, op. cit.; R. Deheuvels, *Formes quadratiques et groupes classiques*, op. cit.; C. Chevalley, *The Algebraic Theory of Spinors*, op.cit.

Thus, any element f belonging to $(S_C)_s$ can be written $f = (\pm \mathrm{Id})_{E_n} \circ f_0$ with $f_0 \in \boxed{C_n(p,q)}$ since $e_0 e_{n+1} e_1$ and $e_0 e_1 e_2$ are odd and since $-\mathrm{Id}_{E_n}$ belongs to $(S_C)_s$, as previously said. We are thus led to the previous demonstration (i).

The results concerning the dimension come from those given[67] for the spinoriality groups. The same method as in 2.4.2.6 leads to the determination of these groups and to the determination of their number (cf. below exercises).

2.5.1.5 Description of the Real Conformal Groups of Spinoriality in a Strict Sense

As for the classical spinoriality groups studied by A. Crumeyrolle[68] normalization conditions appear. We obtain the following statement (cf. exercises below):

S_C is the subgroup of $(C_n(p,q)_r)$ of elements f_1 that stabilize the m.t.i.s. associated with the r-isotropic vector $f_r = y_1 \cdots y_r$ and satisfy

$$f_1(y_1) \cdots f_1(y_r) = \pm y_1 \cdots y_r .$$

In elliptic signature the group S_C has 2 connected components. $\dim S_C = r^2 + 2r$. *If Q is a neutral form $(p = r)$, S_C has 2 connected components if r is even and 4 connected components if r is odd.* $\dim S_C = r(3r+5)/2$. *In signature (p,q), $p \leq n-q$, p positive terms $r > 2$; if pq is even, S_C has 2 connected components and if pq is odd, S_C has 4 connected components.* $\dim S_C = (r+1)^2 - 2 + p(p+1)/2$.

2.5.1.6 Remarkable Factorization of Elements of $(S_C)_e$ and S_C and Topological Remarks if $n = 2r$

If pq is even, any element $f_0 \in (S_C)_e$ can be written in the form $f_0 = \Omega \circ \mathcal{T} \circ \mathcal{H} \circ S$, where $\Omega \in \boxed{O(p,q)}$ and stabilizes $F'_r = \{y_1, \ldots, y_r\}$ and therefore belongs to σ_e, the classical spinoriality group associated with f_r (cf. below 3.10.1.5). $\mathcal{T}$ is a translation, $\mathcal{H}$ is dilation, and S is a special conformal transformation: $x \to x(1+ax)^{-1}$.

If pq is odd, f_0, belonging to $(S_C)_e$ ca be written $f_0 = (\pm Id_{E_n}) \circ \Omega \circ \mathcal{T} \circ \mathcal{H} \circ S$ with Ω belonging to σ_e.

Thus, we obtain that

$$\frac{\boxed{(S_C)_e}}{\boxed{\sigma_e}}$$

is homeomorphic to $\mathbf{R}^{2n+1}$, taking account of the topological remark already used in 2.4.6.2. (The group of the translations of E_n has n parameters, $\mathbf{R}^+$ is homeomorphic

[67] A. Crumeyrolle, *Fibrations Spinorielles et Twisteurs Généralisés*, op. cit.

[68] Idem.

to $\mathbf{R}$. Finally we recall that $\boxed{C_n(p,q)}$ is homeomorphic to $\boxed{O(p,q)}$ $\times\, \mathbf{R}^{2n+1}$.)

We will use this remark later in 2.7.2.2.

One can easily extend such a factorization to the case of the real conformal spinoriality group S_c in a strict sense.

2.5.2 Flat Conformal Spin Structures in Even Dimension

2.5.2.1 Witt Decomposition

Let $\mathbf{R}^{2r}$ be endowed with a quadratic form of signature (p,q): we suppose that $p \leq n-p$, $(n = 2r)$. We introduce[69] a "real" "special Witt decomposition" of $\mathbf{C}^{n+2} = E'_{n+2} = (E_{n+2})_{\mathbf{C}}$, naturally associated with the previous basis of E_{n+2} : $\{e' \ldots, e_n, e_0, e_{n+1}\} : (W_1)_{n+2} = \{x_i, y_j\}$ with

$$\begin{aligned}
x_1 &= \frac{e_1+e_n}{2}, \ldots, x_p = \frac{e_p+e_{n-p+1}}{2}, x_{p+1} = \frac{ie_{p+1}+e_{n-p}}{2}, \ldots,\\
x_r &= \frac{ie_r+e_{n-r+1}}{2}, x_0 = \frac{e_0+e_{n+1}}{2}\\
y_1 &= \frac{e_1-e_n}{2}, \ldots, y_p = \frac{e_p-e_{n-p+1}}{2}, y_{p+1} = \frac{ie_{p+1}-e_{n-p}}{2}, \ldots,\\
y_r &= \frac{ie_r-e_{n-r+1}}{2}, y_0 = \frac{e_0-e_{n+1}}{2}.
\end{aligned}$$

So that for all i and j, $B(x_i, y_i) = \delta_{ij}/2$ and $x_i y_j + y_j x_i = \delta_{ij} = 2B(x_i, y_i)$, $0 \leq i \leq r$, $0 \leq j \leq r$. We know[70] that for each Witt decomposition of E'_{n+2}, $E'_{n+2} = F + F'$, we can find a basis of isotropic vectors $\{\eta_1, \ldots, \eta_r, \eta_0\}$ in F' and a basis of isotropic vectors $\{\xi_1, \ldots, \xi_r, \xi_0\}$ in F such that $\{\xi_i, \eta_j\}$ is a "real" Witt basis of E'_{n+2}. With the same notation as 2.5.1.1, we consider $\eta = k_1 \circ \tilde{\varphi}$ from $RO(p+1, q+1)$ onto $PO(p+l, q+1)$ via the exact sequence

$$1 \longrightarrow \mathcal{A} \longrightarrow RO(p+1,q+1) \xrightarrow{\eta} PO(p+1,q+1) \longrightarrow 1$$

and $\eta' = k'_1 \circ \tilde{\varphi}'$, so that we have the corresponding exact sequence

$$1 \longrightarrow \mathcal{A}' \longrightarrow RO'(n+2) \xrightarrow{\eta'} PO'(n+2) \longrightarrow 1.$$

Let Cl'_{n+2} be the complexified algebra of Cl_{n+2}, and let ϱ be the classical spin representation[71] of Cl'_{n+2} corresponding to the left action of Cl'_{n+2} on the minimal ideal $Cl'_{n+2} f_{r+1}$ (where $f_{r+1} = y_1 y_2 \cdots y_r y_0$ is an isotropic $(r+1)$-vector), called[72] "the space of conformal spinors" associated with $E_n(p,q)$.

[69] P. Anglès, *Les Structures Spinorielles Conformes Réelles*, op. cit., p. 40.
[70] A. Crumeyrolle, *Fibrations Spinorielles et Twisteurs Généralisés*, op. cit.
[71] C. Chevalley, *The Algebraic Theory of Spinors*, op. cit.
[72] Cf. P. Anglès, Les Structures spinorielles conformes réelles, op. cit.

2.5.2.2 General Definitions

We will use the definitions recalled in 1.2.2.8.4. In particular φ, mentioned in 2.5.3.1, denotes the spin representation. We consider[73] the projective space $P(E'_{n+2})$ and *projective Witt frames of* $P(E'_{n+2})$ associated with Witt basis of F'_{n+2} and in particular *projective orthogonal Witt frames of* $P(E'_{n+2})$.

Let $(\tilde{\Omega}_{n+2})_1$ and $(\tilde{W}_{n+2})_1$ be two projective orthogonal Witt frames of $P(E'_{n+2})$ so that $(\tilde{W}_{n+2})_1 = \tau_1^{-1}(\tilde{\Omega}_{n+2})_1$, where $\tau_1^{-1} \in PO(p+1, q+1)$. Classically,[74] we identify the complexification of τ_1 with τ_1. Thus, we determine the action of $RO(p+1, q+1)$ on $PO(p+1, q+1)$. Let g be one of the four elements of $RO(p+1, q+1)$ such that $\eta(g) = \tau_1 \in PO(p+1, q+1)$. We observe that $\tau_1 = \eta(g) = \eta(-g) = \eta(e_N g) = \eta(-e_N g)$.

If $(\tilde{W}_{n+2})$ is a projective Witt frame of $P(E'_{n+2})$ associated with an orthogonal projective frame of $P(E'_{n+E})$ and with a "real" orthonormal basis $(\mathcal{B}'_1)_{n+2}$ of E'_{n+2} and with a "real" orthonormal basis $(\mathcal{B}'_1)$ of E'_n (e_0, e_{n+1} being chosen once and for all), *we define*,[75] "over" the *orthonormal "real" basis* $(\mathcal{B}'_{1n})$ *of* E'_n *the four spinor frames called conformal spinor frames or* E_n:

$$\{\varepsilon_1(x_{i_0}x_{i_1}\cdots x_{i_h}f_{r+1})\}, \quad \text{where } \varepsilon_1 = \pm 1 \text{ or } \pm e_N, \qquad i_0 < i_1 < \cdots < i_h,$$

such that if $\eta(g) = \tau_1 \in PO(p+1, q+1)$ and if $\delta \in \{g, -g, ge_N, -ge_N\}$ we have

$$x_{i_0}x_{i_1}\cdots x_{i_h}f_{r+1} = \delta^{-1}\xi_{i_0}\xi_{i_1}\cdots\xi_{i_h}\delta f_{r+1} = \varrho(\delta^{-1})\xi_{i_0}\xi_{i_1}\cdots\xi_{i_h}\delta f_{r+1}.$$

This is equivalent to

$$\varrho(\delta)[x_{i_0}x_{i_1}\cdots x_{i_h}f_{r+1}] = \delta x_{i_0}x_{i_1}\cdots x_{i_h}f_{r+1} = \xi_{i_0}\xi_{i_1}\cdots\xi_{i_h}\delta f_{r+1}.$$

Thus, $(\tilde{\mathcal{R}}_{n+2})_1 = \eta(\delta)(\tilde{\mathcal{R}}'_{n+2})_1$ is equivalent to $S_{n+2} = \varrho(\delta)S'_{n+2}$, where $(\tilde{\mathcal{R}}_{n+2})_1$ and $(\tilde{\mathcal{R}}'_{n+2})_1$ (respectively, S_{n+2} and S'_{n+2}) are projective orthogonal frames in the projective space $P(E'_{n+2})$, respectively "conformal spinor frames" with $S'_{n+2} = x_{i_0}\cdots x_{i_h}f_{r+1}$ and $S_{n+2} = \xi_{i_0}\xi_{i_1}\cdots\xi_{i_h}\delta f_{r+1}$.

2.5.2.2.1 Definition A conformal spinor of E_n, associated with a complex representation ϱ of $RO(p+1, q+1)$ in a space of spinors for the Clifford algebra Cl'_{n+2}, is by definition an equivalence class $((\tilde{\mathcal{R}}_{n+2})_1, g, \chi_{n+2})$, where $(\tilde{\mathcal{R}}_{n+2})_1$ is a projective orthogonal frame of $P(E'_{n+1})$, $g \in RO(p+1, q+1)$, $\chi_{n+2} \in \mathbf{C}^{2^{r+1}}$ and where $((\tilde{\mathcal{R}}'_{n+2})_1, g', \chi'_{n+2})$ is equivalent to $((\tilde{\mathcal{R}}_{n+2})_1, g, \chi_{n+2})$ if and only if we have $((\tilde{\mathcal{R}}'_{n+2})_1 = \sigma((\tilde{\mathcal{R}}_{n+2})_1)$, $\sigma = \eta(\gamma) \in PO(p+1, q+1)$ with $\gamma = g'g^{-1}$ and $\chi'_{n+2} = {}^t(\varrho(\gamma))^{-1}\chi_{n+2}$, where ${}^t(\varrho)^{-1}$ is the dual representation of ϱ and where $(\varrho(\gamma))^{-1}$ is identified with an endomorphism of $\mathbf{C}^{2^{r+1}}$.

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[73] Idem.

[74] A. Crumeyrolle, *Structures Spinorielles*, op. cit.

[75] P. Anglès, (a) Construction de revêtements du groupe conforme d'un espace vectoriel muni d'une 4métrique* de type (p, q), op. cit.; (b) Les structures spinorielles conformes réelles, op. cit.

We can also write $(\tilde{\mathcal{R}}'_{n+2})_1 = (\tilde{\mathcal{R}}_{n+2})_1\sigma$ instead of $(\tilde{\mathcal{R}}'_{n+2})_1 = \sigma(\tilde{\mathcal{R}}_{n+2})_1$, which defines a right action, and in the same way, we can use the associated projective orthogonal Witt frames of $P(E'_{n+2}) : (\tilde{\Omega}_{n+2})_1, (\tilde{\Omega}'_{n+2})_1$.

2.5.2.2.2 Definition We agree to call by definition an equivalence class $((\tilde{\mathcal{R}}_{n+2})_1, g)$, where g is in $RO(p+1, q+1)$ and $(\tilde{\Omega}_{n+2})_1$ is a projective orthogonal frame of $P(E'_{n+2})$ a conformal spinor frame of E_n associated with the "real" orthonormal basis $(\mathcal{B}_1)_n$ of E'_n. $((\tilde{\mathcal{R}}_{n+2})_1, g)$ is equivalent to $((\tilde{\mathcal{R}}'_{n+2})_1, g')$ if and only if $(\tilde{\mathcal{R}}'_{n+2})_1 = (\tilde{\mathcal{R}}_{n+2})_1\sigma$ and $\sigma = \eta(\gamma)$ with $g, g' \in RO(p+1, q+1)$, and $\gamma = g'g^{-1}$.

We remark that

$$((\tilde{\mathcal{R}}_{n+2})_1, g) \sim ((\tilde{\mathcal{R}}_{n+2})_1, -g) \sim ((\tilde{\mathcal{R}}_{n+2})_1, e_N g) \sim ((\tilde{\mathcal{R}}_{n+2})_1, -e_N g).$$

If we suppose $g, g' \in RO'(n+2)$ *with* $\gamma = g'g^{-1} \in RO(p+1, q+1)$, *we can consider the action of* $RO(p+1, q+1)$ *on every spinor frame of* $Cl'_{n+2} f_{r+1}$.

2.5.2.2.3 Definition With obvious notation, $(\tilde{\Omega}_{n+2})_1$ and $(\tilde{\Omega}'_{n+2})_1$ being projective orthogonal Witt frames of $P(E'_{n+2})$, $((\tilde{\Omega}_{n+2})_1, g)$ and $((\tilde{\Omega}'_{n+2})_1, g')$ *define the same flat conformal spin structure if and only if* $(\tilde{\Omega}'_{n+2})_1 = \sigma(\tilde{\Omega}_{n+2})_1$, $\eta'(\gamma) = \sigma$, $\gamma = g'g^{-1}$, $g, g' \in RO'(n+2)$, $\gamma \in RO(p+1, q+1)$.

(Thus $((\tilde{\Omega}_{n+2})_1, g) \sim ((\tilde{\Omega}_{n+2})_1, -g) \sim ((\tilde{\Omega}_{n+2})_1, e_N g) \sim ((\tilde{\Omega}_{n+2})_1, -e_N g)$.) We define[76] complex conformal spin flat structures, using the mapping η' from $\mathrm{Pin}'(n+2)$ onto $PO'(n+2)$ with kernel $\mathcal{A}'$.

2.5.3 Case $n = 2r+1, r > 1$

2.5.3.1 Definitions

If in an orthonormal basis of E_n we can write $q(x) = (x^1)^2 + \cdots + (x^p)^2 - (x^{p+1})^2 - \cdots - (x^{2r})^2$, under the assumption that $p \leq 2r - p$, we obtain[77] a Witt decomposition of $E'_n = F + F' + \{e_n\}$, where F and F' are defined as previously for the case $n = 2r$.

If we consider a special Witt basis $\Omega_n = \{\xi_i, \eta_j, e_n\}$ and $W_n = \{x_i, y_j, z_n\}$ associated with a real orthonormal basis of E'_n, according to the fact that $e_N = e_0 e_{n+1} e_1 \cdots e_n$ belongs to the center of the Clifford algebra $Cl(E_{n+2})$ (we recall that the center of $Cl(E_{n+2})$ is then $\mathbf{R} \oplus \mathbf{R}e_N$; cf. Chapter 1), we can define real conformal spinorial frames $\{\varepsilon_1 S\}$ where $S = \{x_{i_0} x_{i_1} \cdots x_{i_h} f_{r+1}\}$, $i_0 < i_1 < \cdots < i_h$, with $\varepsilon_1 = \pm 1$ or $\pm e_N$ for $Cl'^{+}(Q_1)_{n+2}) f_{r+1}$,[78] so that $S_{n+2} = \rho(\delta) S'_{n+2}$ is equivalent to

76 P. Anglès, Les structures spinorielles conformes réelles, Thesis, op. cit., cf. below, exercises.

77 A. Crumeyrolle, *Structures Spinorielles*, op. cit.

78 If $n = 2r+1$, then $Cl^{+}(E_n)$ is central simple. If Q is of maximal index r, $Cl^{+}(E_n)$ is isomorphic to $Cl(Q_1)$, where $Cl(Q_1)$ is the Clifford algebra for the space $F + F'$, where

$(\tilde{\mathcal{R}}_{n+2})_1 = \eta(\delta)(\tilde{\mathcal{R}}'_{n+2})_1$, $\delta \in RO^+(p+1, q+1)$ or $\delta \in RO^{+'}(n+2)$ with previous notation.

We can consider $RO^+(p+1, q+1)$ or $RO^{+'}(n+2)$ since we represent only Cl'^{+}_{n+2}. We can extend previous definitions, taking account of these remarks. We replace Ω_n by $\{\xi_i, \eta_j, e_n\}$ and W_n by $\{x_i, y_j, z_n\}$.

The same is done for previous orthogonal projective frames and Witt projective frames.

2.5.3.2 Remark

Since $n = 2r + 1$, we now obtain that $e_N f_{r+1} = f_{r+1} e_N = (-i)^{r-p} f_{r+1}$ since e_N is in the center of the corresponding Clifford algebra.

2.5.3.3 Conformal Spinoriality Groups

One can easily define conformal spinoriality groups associated with Cl'^{+}_{n+2} as previously. Since we only represent Cl'^{+}_{n+2}, we find again that $C_n(p, q)$ isomorphic to $PO(p+1, q+1)$ is isomorphic to $\psi(RO^+(p+1, q+1)) = SO(p+1, q+1)$. We recall that it is known[79] that if $n = 2r + 1$, $PO(p+1, q+1)$ is isomorphic to $SO(p+1, q+1)$.

2.6 Real Conformal Spin Structures on Manifolds

2.6.1 Definitions

V is a real paracompact n-dimensional pseudo-riemannian (in particular, riemannian) manifold. Its fundamental tensor field is called, abusively, Q. We denote by $\xi(E, V, O(p, q), \pi)$, or simply ξ, the principal bundle of orthonormal frames of V (If n odd, $n = 2r + 1$, we assume that M is orientable.)

2.6.1.1 Bundle $\xi_1(V)$

Let $i : O(p, q) \to C_n(p, q)$ be the canonical injective homomorphism. The group $O(p, q)$ acts on $C_n(p, q)$ by $(\omega, f) \in O(p, q) \times C_n(p, q) \to i(\omega) f \in C_n(p, q)$.

$F + F'$ is such that $F + F' + (\xi_0)$ is a Witt decomposition of E_n, with ξ_0 being nonisotropic, and where Q_1 is related to Q by $Q_1(y) = -Q(\xi_0)Q(y)$, $y \in F + F'$. Q_1 is a neutral form and $Cl(Q_1)$ can be represented as $Cl(E_n)$ for $n = 2r$. $Cl^+(E_n)$, for $n = 2r + 1$, possesses a spinorial representation ρ_+ which can be extended exactly in two inequivalent ways to an irreducible representation ρ of $Cl(E_n)$. (Cf. chapter 1 of C. Chevalley, The algebraic theory of spinors. op.cit.)

[79] Cf. J. Dieudonné, On the automorphisms of the classical groups, *Mem. Amer. Math. Soc.*, no. 2, 1951, pp. 1–95.

Let $\xi_1(A_1, V, C_n(p,q), \omega_1)$ be the principal bundle with structure group $C_n(p,q)$ over the same base V, obtained by i-extension of ξ.[80] $\xi_1(V) = i(\xi(V)) = \xi_i(V) = \xi_1(A_1, V, C_n(p,q), \bar{\omega}_1)$ is a principal bundle with structure group $C_n(p,q)$ in the following way: let us choose a covering $(\mathcal{U}_{\alpha'})_{\alpha' \in A}$ of V with a system of local cross sections $\sigma_{\alpha'}$ and transition functions $g_{\alpha'\beta'}$. Let us define maps $g'_{\alpha'\beta'} = i \circ g_{\alpha'\beta'}$. Then, for all $x \in U_{\alpha'} \cap U_{\beta'} \cap U_{\gamma'}$ the $g'_{\alpha'\beta'}$ satisfy the relation $g'_{\alpha'\beta'}(x) g'_{\beta'\gamma'}(x) = g'_{\alpha'\gamma'}(x)$ and consequently, there is a principal bundle ξ_i with a system of local sections such that the $g'_{\alpha'\beta'}$ are the corresponding transition functions, according to a general result of Greub and Petry.[81]

2.6.1.2 Bundle $P\xi_1(V)$

Let us recall that $C_n(p,q)$ is isomorphic to $PO(p+1, q+1)$. Using, with previous notation, the classic sequence of groups

$$1 \longrightarrow \mathbf{Z}_2 \longrightarrow O(p+1, q+1) \xrightarrow{\tilde{h}} PO(p+1, q+1) \longrightarrow 1$$

(cf. 2.5.1.2), let us define $\tilde{\lambda} = \tilde{h} \circ j = k_1 \circ i$ and let $P\xi_1(V) = \tilde{\lambda}(\xi(V)) = \xi_{\tilde{\lambda}}(V)$ be the $\tilde{\lambda}$-extension of the principal bundle $\xi(V)$. $P\xi_1(V) = \xi_{\tilde{\lambda}}(V) = P\xi_1(E'_1, V, PO(p+1, q+1), \pi_1)$ is a principal bundle with structure group $PO(p+1, q+1)$ over the same base V. Thus, e_0 and e_{n+1} being chosen once and for all, the two bundles ξ_1 and $P\xi_1$ are isomorphic. Subsequently, since the action of $PO(p+1, q+1)$ on the set of projective frames of $P(E_{n+2})$ is simply transitive, it is suitable to retain $P\xi_1$ the principal bundle, $\tilde{\lambda}$-extension of ξ, with structure group $PO(p+1, q+1)$.

2.6.1.3 Bundle $Clif_1(V)$

Let us introduce $\theta(V)$ the trivial bundle with typical fiber $\mathbf{R}^2$ with a quadratic form Q_2 of signature $(1,1)$, and let us write $\theta(V) = \xi_0 \oplus \xi_{n+1}$, since a Whitney sum of two bundles with typical fiber $\mathbf{R}$ and the required condition of orthogonality for Q_2.

We define then $T_1(V) = T(V) \oplus \theta(V) = \cup_{x \in V} T_1(x)(V)$, where $T(V)$ is the tangent bundle of V and $T_1(x)(V) = T(x) \oplus (\xi_0)_x \oplus (\xi_{n+1})_x$, with obvious notation.

We denote by $Clif(V, Q)$ or simply $Clif(V)$, the Clifford bundle of V, and we introduce another bundle $Clif_1(V)$ in the following way. At any point $x \in V$, let us consider $\otimes T_1(x)$ and the Clifford algebra $(Cl_{n+2})_x$ obtained as a quotient algebra of $\otimes T_1(x)$ by the ideal generated by $X_1(x) \otimes X_1(x) - Q_{n+2}(X_1(x))$, where $X_1(X) \in T_1(X)$ and Q_{n+2} is the quadratic form of signature $(p+1, q+1)$ defined on $\mathbf{R}^{n+2}$.

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80 W. Greub and R. Petry, On the lifting of structure groups, Lecture Notes in Mathematics, no 676. *Differential Geometrical Methods in Mathematical Physics*, Proceedings, Bonn, 1977, pp. 217–246.

81 Idem.

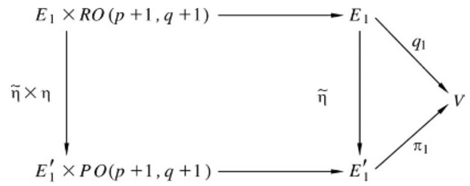

Fig. 2.2.

The collection of the Clifford algebras $(Cl_{n+2})_x$ is naturally a vector bundle of typical fiber $Cl_{n+2}(p+1, q+1)$, which we denote by $Clif_1(V)$ and which is an "amplified Clifford bundle" in the same way as $T_1(V)$ is an "amplified tangent bundle." It is possible to define the action of the group $C_n(p, q)$ on such a bundle by means of the representation K_1 so settled. For any ω belonging to $Cl_{n+2}(p + 1, q + 1)$, for any $\varphi(g) \in C_n(p, q)$ we set $K_1\varphi(g)\omega = \pi(g)\omega g^{-1}$, which defines a representation of $C_n(p, q)$ into $Cl_{n+2}(p + 1, q + 1)$. Thus, $PO(p + 1, q + 1)$ isomorphic to $C_n(p, q)$ acts on $Clif_1(V)$. $Clif'_1(V)$ denotes its complexification and in the same way as previously, we can define the action of $PO'(n + 2)$ isomorphic to C'_n on this bundle.

2.6.2 Manifolds of Even Dimension Admitting a Real Conformal Spin Structure in a Strict Sense

Let V be a real paracompact n-dimensional smooth pseudo-riemannian (in particular riemannian) manifold. In this paragraph and the next three we assume that n is even, $n = 2r$. As in 1, ξ stands for the bundle of orthonormal frames of V; $P\xi_1(E'_1, V, PO(p+1, q+1), \pi_1)$ is the principal bundle obtained as the $\tilde{\lambda}$-extension of ξ. We agree to give the following definitions, which generalize those given by A. Crumeyrolle[82] for the orthogonal case to the conformal orthogonal one.

2.6.2.1 Definition V admits a real conformal spin structure in a strict sense if there exists a principal fiber bundle $S_1(E_1, V, RO(p + 1, q + 1), q_1)$ and a morphism of principal bundles $\tilde{\eta} : S_1 \to P\xi_1$ such that S_1 a 4-fold covering of $P\xi_1$ with the following commutative diagram (see Figure 2.2), where the horizontal mappings correspond to right translations. S_1 is called the bundle of conformal spinor frames of V.

2.6.2.2 Definition According to this definition, we introduce the bundle of conformal spinors

$$\sigma_1 = \left(\frac{S_1(V) \times \mathbf{C}^{2^{r+1}}}{RO(p+1, q+1)}, V, RO(p+1, q+1), \mathbf{C}^{2^{r+1}} \right),$$

[82] A. Crumeyrolle, Structures spinorielles, *Ann. Inst. H. Poincaré, Section A (N.S.)*, 11, 1969, pp. 19–055. A. Lichnerowicz, (a) Champs spinoriels et propagateurs en relativité générale, *Bull. Soc. Math. France*, 92, 1964, pp. 11–100; (b) Champ de Dirac, champ du neutrino et transformations C. P. T. sur un espace-temps courbe, *Ann. Inst. H. Poincaré, Section A (N.S.)*, 1, 1964, pp. 233–290.

a complex vector bundle of dimension 2^{r+1} with typical fiber $\mathbf{C}^{2^{r+1}}$ associated with the bundle $S_1(V)$ of *"conformal spinor frames."* We write $\sigma_1 = (\sigma_1', V, RO(p+1, q+1), s_1)$.

Remarks. It is always possible to define the two fibrations $P\xi_1$ and S_1 by means of the same trivializing neighborhoods $(U_{\alpha'})_{\alpha' \in A}$ and local cross section $z_{\alpha'}, \tilde{\mathcal{R}}_{\alpha'}$ with transition functions, $\gamma_{\alpha',\beta'}$ respectively $\eta(\gamma_{\alpha',\beta'})$:

$$\begin{aligned} z_{\beta'}(x) &= z_{\alpha'}(x)\gamma_{\alpha',\beta'}(x), \quad \gamma_{\alpha',\beta'}(x) \in RO(p+1, q+1), \\ \tilde{\eta}(z_{\beta'}(x)) &= \tilde{\mathcal{R}}_{\beta'}(x) = \tilde{\eta}(z_{\alpha'}(x))\eta(\gamma_{\alpha',\beta'}(x)) \\ &= \tilde{\mathcal{R}}_{\alpha'}(x)\eta(\gamma_{\alpha',\beta'}(x)), \quad \eta(\gamma_{\alpha',\beta'}) \in PO(p+1, q+1). \end{aligned}$$

$\tilde{\mathcal{R}}_{\alpha'}(x)$ and $\tilde{\mathcal{R}}_{\beta'}(x)$ are "projective orthogonal frames" of $P(E'_{n+2})$.

Let us consider the Clifford algebra Cl_{n+2} of $E_{n+2}(p+1, q+1)$ and the complexified algebra Cl'_{n+2} isomorphic to $Cl_{n+2}(Q')$, where Q' is the complexification of Q.[83] The sequence

$$\{x_{i_0}x_{i_1}\cdots x_{i_h}y_{j_0}y_{j_1}\cdots y_{j_h}\} \begin{cases} 0 \le h \le r, \\ 0 \le i_0 < i_1 < i_2 \cdots \le r, \\ 0 \le j_0 < j_1 < j_2 \cdots \le r, \end{cases}$$

is a basis of Cl'_{n+2}, where $\{x_i, y_j\}$ is the special Witt basis of $\mathbf{C}^{n+2}$; already used. *The choice of the above basis establishes a linear isomorphism μ between Cl'_{n+2} and $\mathbf{C}^{2^{n+2}}$.*

We can observe that the spinorial bundle σ_1 associated with the bundle S_1 is a principal bundle with typical fiber $\mathbf{C}^{2^{r+1}}$ and structure group $RO(p+1, q+1)$, which acts effectively in $\mathbf{C}^{2^{r+1}}$ ($\mathbf{C}^{2^{r+1}}$ is an irreducible Cl'_{n+2}-representation space).

It is permissible to choose any irreducible representation of Cl'_{n+2} in $\mathbf{C}^{2^{r+1}}$ and convenient to choose the representation corresponding to the left action of Cl'_{n+2} in the minimal ideal of conformal spinors, $Cl'_{n+2}f_{r+1} = Cl'_{n+2}y_1y_2\cdots y_ry_0$, of which the $\{\varepsilon_1 x_{i_0}x_{i_1}\cdots x_{i_h}f_{r+1}\}$ where $\varepsilon_1 = \pm 1$ or $\pm e_N$ constitute "*four conformal spinor frames*" (cf. 2.6.2 and 2.5.2). By restriction of μ to $Cl'_{n+2}f_{r+1}$ we obtain a linear identification of $Cl'_{n+2}f_{r+1}$ *with* $\mathbf{C}^{2^{r+1}}$.

Over an open set of V, endowed with the cross section $z : x \to z(x)$ of S_1 a conformal spinor field χ will be defined by a differentiable mapping χ from E_1 into $\mathbf{C}^{2^{r+1}} : z \to \chi(z)$ such that[84] if $\chi(z) = \mu(u)$, $u \in Cl'_{n+2}f_{r+1}$,$(u = vf_{r+1})$, then $\chi(z\gamma^{-1}) = \gamma\chi(z) = \mu(\gamma u)$, $(\forall\, \gamma)$, $(\gamma \in RO(p+1, q+1))$. We denote by χ_x the restriction of χ to $\mathcal{S}_{1x} = s_1^{-1}(x)$ and observe that

$$(\chi_x(z))^{i_0 i_1\cdots i_h}x_{i_0}x_{i_1}\cdots x_{i_h}f_{r+1} = (\chi_x(z))^{i_0 i_1\cdots i_h}(\gamma x_{i_0}x_{i_1}\cdots x_{i_h})f_{r+1}. \qquad \text{(I)}$$

[83] P. Anglès, Les structures spinorielles conformes réelles, Thesis, op. cit.

[84] A. Lichnerowicz, Champs spinoriels et propagateurs en relativité générale, op. cit.

2.6.3 Necessary Conditions for the Existence of a Real Conformal Spin Structure in a Strict Sense on Manifolds of Even Dimension

Let $x \to z_x$ be a local cross section over $\mathcal{U}$, a trivializing open set in the bundle S_1. We set

$$z_x = v(x, g(x)) = v^x(g(x)), \quad g(x) \in RO(p+1, q+1)$$

according to the construction of associated bundles $[z_x, x_{i_0} \cdots x_{i_h} f_{r+1}]$, identified to $[z_x \gamma^{-1}, \gamma^x_{(i)} f_{r+1}]$, is a cross section over $\mathcal{U}$ in the bundle σ_1 which we denote by $[z_x, x_{(i)} f_{r+1}]$ or $M^x(x_{(i)} f_{r+1})$. Let also $\tilde{\mathcal{R}}_x = \tilde{\eta}(z_x)$.

Let $(\mathcal{U}_{\alpha'})_{\alpha' \in A}$ be a trivializing atlas for the bundle $P\xi_1$. We can always suppose that there exists over $(\mathcal{U}_{\alpha'})$ a cross section $z_{\alpha'}$ in S_1; we take again $\tilde{\mathcal{R}}_{\alpha'}(x) = \tilde{\eta}(z_{\alpha'}(x))$. If $\tilde{W}_{\alpha'}(x)$ is the projective "real" Witt frame associated with the projective orthogonal frame $\tilde{\mathcal{R}}_{\alpha'}(x)$, we write, *abusively*, $\tilde{\eta}(z_{\alpha'}(x)) = \tilde{W}_{\alpha'}(x)$. In the projective space $P(\mathbf{C}^{n+2})$ the projective "real" frame

$$\underbrace{\{p(x_0), \ldots, p(x_r), p(y_0), \ldots, p(y_r), p(x_0 + \cdots + x_r + y_0 + \cdots + y_r)\}}_{(2r+3)\text{ elements}}$$

corresponds to the "real" Witt basis $\{x_i, y_j\}$, $0 \le i \le r, 0 \le j \le r$ of $\mathbf{C}^{n+2}$ (p is the canonical map: $E_{n+2} \to P(E_{n+2})$). We agree to denote such a projective "real" frame by $\widetilde{\{x_i, y_j\}}$. Since the action of $PO(p+1, q+1)$ on the set of projective orthogonal frames is simply transitive, we can write

$$\tilde{W}_{\alpha'}(x) = \tilde{\eta}(z_{\alpha'}(x)) = \tilde{\Theta}^x_{\alpha'}(\widetilde{\{x_i, y_j\}}),$$

where the $\tilde{\Theta}^x_{\alpha'}$ admit the transition functions $\eta(\gamma_{\alpha'\beta'})$ in $PO(p+1, q+1)$.

If there exists over V a real conformal spin structure in a strict sense, this structure induces in the "amplified" tangent space $T_1(x)$ at x a flat real conformal spin structure (in a purely algebraic way, see 2.5.2) defined by an equivalence class of $(\tilde{\Omega}_x, g_x)$, $g_x \in RO'(n+2)$, $\tilde{\Omega}_x$, a "projective Witt frame," depending differentially on x. Let us recall that

$$(\tilde{\Omega}_x, g_x) \sim (\tilde{\Omega}_x, -g_x) \sim (\tilde{\Omega}_x, e_N g_x) \sim (\tilde{\Omega}_x, -e_N g_x)$$

(see 2.5.2).

We note that $PO'(n+2)$ acts transitively on the set of "real" or complex projective Witt frames, and that in the above class there will always be "real" projective Witt frames. With previous notation, at $x \in U_{\alpha'} \cap U_{\beta'}$ we must obtain two "equivalent frames," which necessarily determine the same flat real conformal spin structure in the "amplified" tangent space at x: $T_1(x)$,

$$(\tilde{\Omega}^x_{\alpha'} = \tilde{\Theta}^x_{\alpha'}\{\pi(\lambda_{\alpha'})(x)\widetilde{\{x_i, y_j\}}\lambda^{-1}_{\alpha'}(x)\}, g_{\alpha'}(x)),$$

and

$$(\tilde{\Omega}^x_{\beta'} = \tilde{\Theta}^x_{\beta'}\{\pi(\lambda_{\beta'})(x)\widetilde{\{x_i, y_j\}}\lambda^{-1}_{\beta'}(x)\}, g_{\beta'}(x)),$$

$\lambda_{\alpha'}(x), \lambda_{\beta'}(x), g_{\alpha'}(x), g_{\beta'}(x) \in RO'(n+2)$, $\lambda_{\alpha'}, \lambda_{\beta'}$ defined respectively over $U_{\alpha'}$ and $U_{\beta'}$ and $g_{\alpha'}, g_{\beta'}$ over a neighborhood of x included in $U_{\alpha'} \cap U_{\beta'}$ with $(g_{\beta'}g_{\alpha'}^{-1})_x \in RO(p+1, q+1)$ and $\eta(g_{\beta'}(x)) = \eta(g_{\alpha'\beta'}(x)g_{\alpha'}(x))$ with $\eta(g_{\alpha'\beta'})$ denoting the transition functions of $\tilde{\Omega}^x_{\alpha'}, \tilde{\Omega}^x_{\beta'}, g_{\alpha'\beta'}$ with values in $RO(p+1, q+1)$, and $\eta(\alpha(\lambda_{\alpha'})g_{\alpha'\beta'}\lambda_{\beta'}^{-1}) = \eta(\lambda_{\alpha'\beta'})$.

We also set

$$\tilde{\Theta}^x_{\alpha'}(\pi(\lambda_{\alpha'}(x))\widetilde{\{x_i, y_j\}}\lambda_{\alpha'}^{-1}(x) = \tilde{\mu}^x_{\alpha'}(\widetilde{\{x_i, y_j\}}).$$

With the notation of 2.6.2, if $\chi(\tilde{\Omega}^x_{\alpha'}, g_{\beta'}(x)) = \mu(f_{r+1})$ then $\chi(\tilde{\Omega}^x_{\beta'}, g_{\beta'}(x)) = \mu(\varepsilon_1 g_{\alpha'\beta'}^{-1}(x)^{-1} f_{r+1})$, where $\varepsilon_1 = \pm 1$ or $\pm e_N$. Since the spinor thus defined at x is a well determined element in $(Cl'_{n+2} f_{r+1})_x$, $\chi(\tilde{\Omega}^x_{\alpha'}, g_{\alpha'}(x)) = \chi(\tilde{\Omega}^x_{\beta'}, g_{\beta'}(x))$, we deduce

$$f_{r+1} = \varepsilon_2 \varepsilon_1 f_{r+1} g_{\alpha'\beta'}^{-1}(x), \tag{I}$$

where $\varepsilon_2 = \pm 1$ *if* r *is even and* $\varepsilon_2 = 1$ *if* r *is odd.*

As matter of fact, let us recall first that e_N anticommutes with every element of E_{n+2}, and that for $g \in RO(p+1, q+1)$, $e_N g = \pm g e_N$, and that $\pi(g) = \pm g$ for $g \in \text{Pin}(p+1; q+1)$. Moreover, $\tilde{\mu}^x_{\alpha'}$ and $\tilde{\mu}^x_{\beta'}$ satisfy the relation

$$\tilde{\mu}^x_{\beta'}(u) = \tilde{\mu}^x_{\alpha'}(\pi(g_{\alpha'\beta'}(x))u g_{\alpha'\beta'}^{-1}(x))$$

for all $u \in Cl'_{n+2} f_{r+1}$. Consequently,

$$\begin{aligned}\tilde{\mu}^x_{\alpha'}(f_{r+1}) &= \tilde{\mu}^x_{\beta'}(\varepsilon_1 g_{\alpha'\beta'}^{-1}(x) f_{r+1}) = \tilde{\mu}^x_{\alpha'}(\pi(g_{\alpha'\beta'})(x)\varepsilon_1 g_{\alpha'\beta'}^{-1}(x) f_{r+1} g_{\alpha'\beta'}^{-1}(x)) \\ &= \tilde{\mu}^x_{\alpha'}(\varepsilon_1 \pi(g_{\alpha'\beta'}(x)g_{\alpha'\beta'}^{-1}(x) f_{r+1} g_{\alpha'\beta'}^{-1}(x)) = \tilde{\mu}^x_{\alpha'}(\varepsilon_1 f_{r+1} g_{\alpha'\beta'}^{-1}(x)).\end{aligned}$$

Therefore, we obtain (I), noting that via the projective space, there appears the factor $\varepsilon_2 = \pm 1$ corresponding to the ambiguity of sign for homogeneous elements of the Clifford algebra.[85] Using the principal antiautomorphism π of the Clifford algebra and observing that for all g belonging to $RO(p+1, q+1)$, $\tau(g) = g^{-1}N(g)$, and that

$$\tau(e_N) = (-1)^{\frac{(n+1)(n+2)}{2}} e_N = (-1)^{r+1} e_N$$

(for $n = 2r$), furthermore

$$\tau(f_{r+1}) = (-1)^{\frac{r(r+1)}{2}} f_{r+1}$$

and

$$\tau(g_{\alpha'\beta'}^{-1}(x)) = \frac{g_{\alpha'\beta'}(x)}{N(g_{\alpha'\beta'}(x))},$$

[85] By getting to the projective space, the product by $(-\text{Id})$ is permissible and gives a factor $(-1)^{r+1}$ for f_{r+1}, whence $\varepsilon_2 = \pm 1$, if r is even and $\varepsilon_2 = 1$, if r is odd.

we obtain

$$\varepsilon_2 f_{r+1} = \frac{g_{\alpha'\beta'}(x)}{N(g_{\alpha'\beta'}(x))} f_{r+1}\tau(\varepsilon_1).$$

Since $f_{r+1}e_N = (-1)^{r+1}(-i)^{r-p} f_{r+1}$ (see 2.5.1.2.1), we get, if $\varepsilon_1 = e_N$,

$$\varepsilon_2 f_{r+1} N(g_{\alpha'\beta'}(x)) = g_{\alpha'\beta'}(x)(-i)^{r-p} f_{r+1},$$

or equivalently,

$$g_{\alpha'\beta'}(x) f_{r+1} = \varepsilon_2 (i)^{r-p} N(g_{\alpha'\beta'}(x)) f_{r+1},$$

and then in any case,

$$(g_{\alpha'\beta'}(x) f_{r+1}) = \varepsilon_2 \varepsilon N(g_{\alpha'\beta'}(x)) f_{r+1}, \tag{II}$$

where

$\varepsilon = \pm 1$ if $r - p = (0 \text{ or } 2) \pmod 4$, and
$\varepsilon = \pm 1$ or $\pm i$, if $r - p = (1 \text{ or } 3) \pmod 4$.

Thus, $g_{\alpha'\beta'}(x)$ belongs to a subgroup H_C of $RO^+(p+1, q+1)$ that is mapped by η onto a subgroup of $PSO(p+1, q+1)$, the special projective orthogonal group isomorphic to a subgroup S_C called[86] "the conformal spinoriality group S_C" in a strict sense, see 2.5.1.3, associated with the r-isotropic vector $f_r = y_1 \cdots y_r$ (we observe that $\varphi(H_C) = S_C \subset (C_n(p, q))_r$, the restricted conformal group,[87] where $\tilde{\varphi}$ is the "projection" from $RO(p+1, q+1)$ onto $C_n(p, q)$). We note that $\pi(g_{\alpha'\beta'}(x)) = g_{\alpha'\beta'}(x)$ since $H_C \subset RO^+(p+1, q+1)$. It is known[88] that

$$g_{\alpha'\beta'}(x) f_{r+1} = \varepsilon_2 \varepsilon N(g_{\alpha'\beta'}(x)) f_{r+1}$$

implies

$$g_{\alpha'\beta'}(x) f_{r+1} g^{-1}_{\alpha'\beta'}(x) = N(g_{\alpha'\beta'}(x))\varepsilon^2 f_{r+1}, \tag{III}$$

since $(N(g_{\alpha'\beta'}(x)))^2 = 1$ and $\varepsilon_2^2 = 1$, with

$$\varepsilon^2 = (i^{r-p})^2 = (-1)^{r-p} = (e_N)^2 = (-1)^{r+q},$$

for $n = p + q = 2r$. Therefore, we have, applying $\tilde{\mu}^x_{\beta'}$ to f_{r+1}, with the following notations,

$$\tilde{\mu}^x_{\beta'}(f_{r+1}) = \tilde{f}_{\beta'}(x) \quad \text{and} \quad \tilde{\mu}^x_{\alpha'}(g_{\alpha'\beta'}(x)) = \tilde{g}_{\alpha'\beta'}(x),$$

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[86] P. Anglès, Les structures spinorielles conformes réelles, Thesis, op. cit.

[87] P. Anglès, Construction de revêtements du groupe conforme d'un espace vectoriel muni d'une métrique de type (p, q), op. cit.

[88] A. Crumeyrolle, Fibrations spinorielles et twisteurs généralisés, op. cit., p. 158. It is shown there that for any $\gamma \in RO(Q)$ and for any isotropic vector f, the condition $\gamma f = \mu f$, $\mu \in$ **C*** is equivalent to $\gamma f \gamma^{-1} = N(\gamma)\mu^2 f$.

taking account of the fact that g is in $RO(p+1, q+1)$ iff $g = v_1 \cdots v_k, v_1, \ldots, v_k \in E_{n+2}$ with $Q(v_i) = \pm 1$, $1 \leq i \leq k$ (cf. chapter 1),

$$\tilde{f}_{\beta'}(x) = \varepsilon_2 \tilde{g}_{\alpha'\beta'}(x) \tilde{f}_{\alpha'}(x) \tilde{g}^{-1}_{\alpha'\beta'}(x). \tag{IV}$$

Then applying $\tilde{\mu}^x_{\alpha'}$ to the previous relation (III), and observing that $N(\tilde{g}_{\alpha'\beta'}(x)) = N(g_{\alpha'\beta'}(x))$, we obtain

$$\tilde{f}_{\beta'}(x) = \varepsilon_2 (e_N)^2 N(\tilde{g}_{\alpha'\beta'}(x)) \tilde{f}_{\alpha'}(x). \tag{V}$$

We observe that $\eta(g_{\alpha'\beta'}(x))$ are transition functions for cross sections in the complexified bundle $(P\xi_1)_{\mathbf{C}}$ of $P\xi_1$. The cocycle $\eta(\gamma_{\alpha'\beta'})$ that defines $P\xi_1$ and the cocycle $\eta(g_{\alpha'\beta'})$ are cohomologous in $PO'(n+2)$. Thus, we have obtained the following:

2.6.3.1 Proposition *If there exists on V a real conformal spin structure in a strict sense,*

(1) *there exists over V an isotropic $(r+1)$-vector pseudofield modulo a factor ε_2, $\varepsilon_2 = \pm 1$ if r is even, $\varepsilon_2 = 1$ if r is odd, pseudo–cross section in the bundle $Clif'_1(V)$.*

(2) *The group of the principal bundle $P\xi_1$ is reducible in $PO'(n+2)$ to a subgroup isomorphic to S_C—the conformal spinoriality group in a strict sense associated with the r-isotropic vector $f_r = y_1 \cdots y_r$—which is a subgroup of $(C_n(p,q))_r$, the restricted conformal group.*

(3) *The complexified bundle $(P\xi_1)_{\mathbf{C}}$ admits local cross sections over trivializing open sets with transition functions $\eta(g_{\alpha'\beta'})$, $g_{\alpha'\beta'}(x) \in RO^+(p+1, q+1)$ such that if the mappings*

$$\tilde{f}_{\alpha'} : x \in U_{\alpha'} \cap U_{\beta'} \to \tilde{f}_{\alpha'}(x)$$

define locally the previous $(r+1)$-isotropic pseudofield, then

$$\tilde{f}_{\beta'} = (e_N)^2 N(g_{\alpha'\beta'}(x)) \tilde{f}_{\alpha'}(x), \text{ modulo } \varepsilon_2,$$

and $\tilde{f}_{\beta'} = \tilde{g}_{\alpha'\beta'}(x) \tilde{f}_{\alpha'}(x) \tilde{g}^{-1}_{\alpha'\beta'}(x)$ modulo ε_2, where $\varepsilon_2 = \pm 1$ if r is even and $\varepsilon_2 = 1$ if r is odd.

2.6.4 Sufficient Conditions for the Existence of Real Conformal Spin Structures in a Strict Sense on Manifolds of Even Dimension

Let us consider the bundle $P\xi_1$.

2.6.4.1 Proposition *Let $(U_{\alpha'}, \tilde{\mu}_{\alpha'})_{\alpha' \in A}$ be a trivializing atlas for the complexified bundle $(P\xi_1)_{\mathbf{C}}$ on V, with transition functions $\eta(g_{\alpha'\beta'}(x)) \in PO(p+1, q+1)$. If there exists over V an isotropic $(r+1)$-vector pseudofield, modulo a factor $\varepsilon_2 = \pm 1$, if r is even and $\varepsilon_2 = 1$, if r is odd, determined locally by means of $x \in U_{\alpha'} \to \tilde{f}_{\alpha'}(x)$ such that if $x \in U_{\alpha'} \cap U_{\beta'} \neq \emptyset$, we have $\tilde{f}_{\beta'}(x) = \tilde{g}_{\alpha'\beta'}(x) \tilde{f}_{\alpha'}(x) \tilde{g}^{-1}_{\alpha'\beta'}(x)$, modulo ε_2, $\tilde{\mu}^x_{\alpha'}(g_{\alpha'\beta'}(x)) = \tilde{g}_{\alpha'\beta'}(x)$, $\tilde{f}_{\beta'} = (e_N)^2 N(\tilde{g}_{\alpha'\beta'}(x)) \tilde{f}_{\alpha'}(x)$, modulo ε_2, then the manifold V admits a real conformal spin structure in a strict sense.*

All the following algebraic calculations are made modulo ε_2, which we omit for simplicity. We abbreviate $\tilde{f}_{\beta'}(x) = f'$, $\tilde{f}_{\alpha'}(x) = f$, $\tilde{g}_{\alpha'\beta'}(x) = \delta$. Then

$$\left\{ \begin{array}{l} f' = \delta f \delta^{-1} \\ f' = (e_N)^2 N(\delta) f \delta \end{array} \right\} \Rightarrow \delta f \delta^{-1} = (e_N)^2 N(\delta) f \Rightarrow \delta f = (e_N)^2 N(\delta) f \delta,$$

whence we deduce since the intersection of any right minimal ideal with any left minimal ideal is of dimension 1,[89] $\delta f = \tilde{\varepsilon}(x) f, \tilde{\varepsilon}(x) \in \mathbf{C}^*$. Then $(e_N)^2 N(\delta) f \delta\delta^{-1} = \tilde{\varepsilon}(x) f \delta^{-1}$; therefore we obtain

$$f\delta^{-1} = \frac{(e_N)^2 N(\delta) f}{\tilde{\varepsilon}(x)}.$$

Applying the principal antiautomorphism τ to $f\delta^{-1}$ we get

$$\tau(\delta^{-1})\tau(f) = \frac{(e_N)^2 N(\delta)}{\tilde{\varepsilon}(x)} \tau(f),$$

or equivalently,

$$\frac{\delta}{N(\delta)} f = \frac{(e_N)^2 N(\delta)}{\tilde{\varepsilon}(x)} f,$$

since $\tau(\delta^{-1}) = \delta / N(\delta)$ (cf. chapter 1), and since $\tau(f) = (-1)^{r(r+1)/2} f$. Thus

$$\delta f = \tilde{\varepsilon}(x) f = (e_N)^2 \frac{N^2(\delta)}{\tilde{\varepsilon}(x)} f,$$

which gives $(\tilde{\varepsilon}(x))^2 = (e_N)^2$, since $(N(\delta))^2 = 1$, with

$$(e_N)^2 = (-1)^{r-p} = \begin{cases} 1 & \text{if } r - p = 0 \text{ (modulo 2)}, \\ -1 & \text{if } r - p = 1 \text{ (modulo 2)}. \end{cases}$$

Then we obtain $\tilde{\varepsilon}(x) = \pm 1$ if $r - p$ is even and $\tilde{\varepsilon}(x) = \pm i$ if $r - p$ is odd. So, we write $\tilde{\varepsilon}(x) = \tilde{\varepsilon}$ and then $g_{\alpha'\beta'}(x) f_{r+1} = \varepsilon_2 \tilde{\varepsilon} f_{r+1}$, where $\tilde{\varepsilon} = \pm 1$ or $\pm i$ and $g_{\alpha'\beta'}(x) \in H_C$ (with previous notation – cf. 2.5.1.3.1).

If at $x \in U_{\alpha'} \cap U_{\beta'}$, $\tilde{\mu}^x_{\alpha'}(y'_1 y'_2 \cdots y'_r y'_0) = \tilde{f}_{\alpha'}(x)$, we can complete the set of vectors $\{y'_1, y'_2, \ldots, y'_r, y'_0\}$ with $\{x'_1, x'_2, \ldots, x'_r, x'_0\}$, so that $\tilde{\mu}^x_{\alpha'}\{\widetilde{x'_i, y'_j}\}$ and $\tilde{\mu}^x_{\beta'}\{\widetilde{x'_i, y'_j}\}$ constitute Witt projective frames in the complexified bundle $(P\xi_1)_{\mathbf{C}}$, with transition functions $\eta(g_{\alpha'\beta'})$. (This is a consequence of the extension of the Witt theorem to the projective orthogonal classical group and to projective orthogonal frames.) Therefore we shall omit the accents and suppose that

$$\tilde{\mu}^x_{\alpha'}(y_1 y_2 \cdots y_r y_0) = \tilde{f}_{\alpha'}(x).$$

[89] C. Chevalley, *The Algebraic Theory of Spinors*, op. cit., p. 71.

Let us consider over $U_{\alpha'}$ the local cross section in $Clif'_1(V)$:

$$x \to (x_{(i)} f_{r+1})^x_{\alpha'} = \tilde{\mu}^x_{\alpha'}(x_{(i)} f_{r+1}).$$

Since for any $\alpha' \in A$, if $x \in U_{\alpha'} \cap U_{\beta'}$, $\tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x) = \varepsilon_2 \tilde{\varepsilon} \tilde{f}_{\alpha'}(x)$, where $\tilde{\varepsilon} = \pm 1$ if $r - p$ is even and $\tilde{\varepsilon} = \pm i$ if $r - p$ is odd, using the principal antiautomorphism τ of the Clifford algebra, we obtain, modulo ε_2,

$$\tau(\tilde{f}_{\alpha'}(x)\tau(\tilde{g}_{\alpha'\beta'}(x))) = \tilde{\varepsilon}_\beta(\tilde{f}_{\alpha'}(x)),$$

or equivalently, $\tilde{f}_{\alpha'}(x) g^{-1}_{\alpha'\beta'}(x) N(\tilde{g}_{\alpha'\beta'}(x)) = \tilde{\varepsilon} \tilde{f}_{\alpha'}(x)$ modulo ε_2—since $\tau(g) = g^{-1}N(g)$—and then

$$\begin{aligned} \tilde{f}_{\beta'}(x) g^{-1}_{\alpha'\beta'}(x) &= (e_N)^2 N(\tilde{g}_{\alpha'\beta'}(x)) \tilde{f}_{\alpha'}(x) g^{-1}_{\alpha'\beta'}(x) \\ &= (e_N)^2 \frac{N(\tilde{g}_{\alpha'\beta'}(x))}{N(g_{\alpha'\beta'}(x))} \tilde{\varepsilon} \tilde{f}_{\alpha'}(x) \text{ (modulo } \varepsilon_2), \end{aligned}$$

and therefore,

$$\tilde{f}_{\beta'}(x) g^{-1}_{\alpha'\beta'}(x) = (e_N)^2 \tilde{\varepsilon} \tilde{f}_{\alpha'}(x) \text{ (modulo } \varepsilon_2),$$

where $(e_N)^2 \tilde{\varepsilon} = (-1)^{r-p} \tilde{\varepsilon}$. We shall write

$$\tilde{f}_{\beta'}(x) \bar{g}^{-1}_{\alpha'\beta'}(x) = \varepsilon' \tilde{f}_{\alpha'}(x) \text{ (modulo } \varepsilon_2),$$

where $\varepsilon' = \tilde{\varepsilon}$ if $r - p$ is even and $\varepsilon' = -\tilde{\varepsilon}$ if $r - p$ is odd.

Then

$$(x_{(i)} f_{r+1})^x_{\beta'} = \varepsilon' \tilde{g}_{\alpha'\beta'}(x)(x_{(i)} f_{r+1})^x_{\alpha'} \text{ (modulo } \varepsilon_2),$$

where ε' is determined in any case [$(x_{(i)} f_{r+1})^x_{\beta'}$ is known, $(x_{(i)} f_{r+1})^x_{\alpha'}$ is known, and one can find an element of the kernel that gives such a relation].

We can associate differentiably with each x in V a 2^{r+1}-dimensional subspace, in $T_1(x)$ the amplified tangent space at x, such that $\tilde{\mu}^x_{\alpha'}(x_{(i)} f_{r+1}) = (x_{(i)} f_{r+1})^x_{\alpha'}$ and the transition functions of $\tilde{\mu}^x_{\alpha'}$ are $\eta(g_{\alpha'\beta'})$. Therefore we have constructed a spinorial bundle over V, with typical fiber $\mathbf{C}^{2r+1}$.

With the frame $\{x_{(i)} f_{r+1}\}^x_{\alpha'}$, we associate the frame $\tilde{\mu}^x_{\alpha'}\widetilde{\{x_i, x_j\}}$. Then with $\{g_{\alpha'\beta'}(x) x_{(i)} f_{r+1}\}^x_{\alpha'}$ is associated $\tilde{\mu}^x_{\beta'}\widetilde{\{x_i, y_j\}}$. We can determine $\lambda_{\alpha'}(x) \in RO'(n+2)$ such that with the frame $\{\lambda_{\alpha'} x_{(i)} f_{r+1}\}$ is associated the frame $\tilde{\mu}^x_{\alpha'}\{\pi(\lambda_{\alpha'})\widetilde{\{x_i, y_j\}}\lambda^{-1}_{\alpha'}\}$, where $\tilde{\mu}^x_{\alpha'}\{\pi(\lambda_{\alpha'})\widetilde{\{x_i, y_j\}}\lambda^{-1}_{\alpha'}\}$ is a "real" projective Witt frame in $(P\xi_1)_{\mathbf{C}}$. We have obtained a real conformal spin structure in a strict sense, since the $\{x_{(i)} f_{r+1}\}^x_{\beta'}$ are local cross sections of a fiber over principal bundle $P\xi_1$.

2.6.4.2 Remark We can observe that the $g_{\alpha'\beta'}(x)$ are defined modulo $\varepsilon_{1\alpha'\beta'}(x) = \pm 1$ or $\pm e_N$. According to previous results (see 2.5.1.1) any real conformal structure will be obtained from the one associated with the choice of $\varepsilon_{1\alpha'\beta'}$ such that $\varepsilon_{1\alpha'\beta'}$ determine a cocyle with values in $\mathbf{Z}_2 \times \mathbf{Z}_2$ if $(e_N)^2 = 1$, respectively in $\mathbf{Z}_4$ if $(e_N)^2 = -1$.

Therefore the set of conformal spin structures is of the same cardinality as $H^1(V, \mathbf{Z}_2 \times \mathbf{Z}_2)$ *if* $r - p$ *is even, respectively as* $H^1(V, \mathbf{Z}_4)$ *if* $r - p$ *is odd.*

2.6.4.3 Proposition *Let us assume that the structure group of the bundle $P\xi_1$ reduces in $PO'(n+2)$ to a subgroup isomorphic to a conformal spinoriality group S_C in a strict sense; then the manifold V admits a real conformal spin structure in a strict sense.*

If we have transition functions $\eta(g_{\alpha'\beta'})$, $g_{\alpha'\beta'}(x) \in H_C$, according to $g_{\alpha'\beta'}(x) \cdot f_{r+1} = \varepsilon f_{r+1}$ with $\varepsilon = \pm 1$ if $r - p \equiv 0$ or 2 (modulo 4) and $\varepsilon = \pm 1$ or $\pm i$ if $r - p \equiv 1$ or 3 (modulo 4), on account of previous remarks, we get

$$\tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x) = \varepsilon_2 \varepsilon \tilde{f}_{\alpha'}(x)$$

(where $\varepsilon_2 = \pm 1$ if r is even and $\varepsilon_2 = 1$ if r is odd).

Using τ the principal antiautomorphism of the Clifford algebra, since $\tau(g) = g^{-1}N(g)$ for all $g \in RO(p+1, q+1)$, we get successively

$$\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x)N(\tilde{g}_{\alpha'\beta'}(x)) = \varepsilon \tilde{f}_{\alpha'}(x) \text{ (modulo } \varepsilon_2)$$

and

$$\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x) = \frac{\varepsilon \tilde{f}_{\alpha'}(x)}{N(\tilde{g}_{\alpha'\beta'}(x))} \text{ (modulo } \varepsilon_2).$$

Since

$$\tilde{f}_{\beta'}(x) = \tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x) \text{ (modulo } \varepsilon_2),$$

we obtain

$$\begin{aligned}\tilde{f}_{\beta'}(x) &= \tilde{g}_{\alpha'\beta'}(x)\frac{\varepsilon \tilde{f}_{\alpha'}(x)}{N(\tilde{g}_{\alpha'\beta'}(x))} = \frac{\varepsilon}{N(\tilde{g}_{\alpha'\beta'}(x))}\tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x) \\ &= \frac{\varepsilon^2}{N(\tilde{g}_{\alpha'\beta'}(x))}\tilde{f}_{\alpha'}(x) \text{ (modulo } \varepsilon_2).\end{aligned}$$

And then

$$\begin{aligned}\tilde{f}_{\beta'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x) &= \frac{\varepsilon^2}{N(\tilde{g}_{\alpha'\beta'}(x))}\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x) = \frac{\varepsilon^3}{(N(\tilde{g}_{\alpha'\beta'}(x)))^2}\tilde{f}_{\alpha'}(x) \\ &= \varepsilon^3 \tilde{f}_{\alpha'}(x) \text{ (modulo } \varepsilon_2),\end{aligned}$$

(since $(N(\tilde{g}_{\alpha'\beta'}(x)))^2 = 1$), where $\varepsilon^3 = \varepsilon$ if $r - p = 0$ or 2 (modulo 4), and $\varepsilon^3 = \pm\varepsilon$ if $r - p = 1$ or 3 (modulo 4).

Starting with this result we can take up again the proof of Proposition 2.6.4.

Remark We observe that the auxiliary bundle $\Theta(V)$ previously introduced does not occur in such a statement, which is therefore intrinsic, since the conformal spinoriality group is defined only by elements of $E_n(p, q)$ and of its complexification E'_n.

2.6.5 Manifolds of Even Dimension with a Real Conformal Spin Structure in a Broad Sense

Let $(C_n(p,q))_r$ be the restricted conformal group (see 2.5.1.1). Let $f_{r+1} = y_1 y_2 \cdots y_r y_0$ be an isotropic $(r+1)$-vector. The enlarged conformal group of spinoriality $(S_C)_e$ associated with the isotropic $(r+1)$-vector f_{r+1} is the subgroup $\varphi((H_C)_e)$ of $(C_n(p,q))_r$, where $((H_C)_e$ is the subgroup of the elements γ of $RO^+(p+1,q+1)$ such that $\gamma f_{r+1} = \mu f_{r+1}$, $\mu \in \mathbf{C}^*$.

In 2.5.1 we proved that $(S_C)_e$ is the "stabilizer" for the action of $(C_n(p,q))_r$ of the m.t.i.s. associated with the isotropic r-vector $y_1 y_2 \cdots y_r$. (We recall that the abbreviation m.t.i.s. stands for maximal totally isotropic subspace.)

2.6.5.1 Definition V admits a real conformal spin structure in a broad sense if and only if the structural group $PO(p+1,q+1)$ of the principal bundle $P\xi_1$—the $\tilde{\lambda}$-extension of the principal bundle ξ of orthonormal frames of V—is reducible to a subgroup of $PO'(n+2)$ isomorphic to $(S_C)_e$, the enlarged conformal group of spinoriality associated with the isotropic r-vector $y_1 y_2 \cdots y_r$.

According to Proposition 2.6.4.3 such a definition is a generalization of definitions given in 2.6.2.

2.6.5.2 Proposition *V admits a real conformal spin structure in a broad sense if and only if there exists over V an $(r+1)$-m.t.i.s. field, that is, a subbundle of $T_1^{\mathbf{C}}(V)$ such that with the same notation as in Proposition 2.6.4.1 we have*

$$\tilde{f}_{\beta'}(x) = \tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x),$$

modulo $\varepsilon_2 = \pm 1$ if r is even, $\varepsilon_2 = 1$ if r is odd, $g_{\alpha'\beta'}(x) \in RO(p+1,q+1)$, $\tilde{f}_{\beta'}(x) = \mu_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)$, $\mu_{\alpha'\beta'}(x) \in \mathbf{C}^*$.

As in the proof of Proposition 2.6.4.1 we obtain $g_{\alpha'\beta'}(x) f_{r+1} = \lambda_{\alpha'\beta'}(x) f_{r+1}$, $\lambda_{\alpha'\beta'}(x) \in \mathbf{C}^*$. Then, taking up again the method given in the proof of Proposition 2.6.4.1 above, we get the result.

Conversely, if it is possible to reduce the structure group $PO(p+1,q+1)$ to a subgroup isomorphic to $(S_C)_e$ in $PO'(n+2)$, the same method as in the proof of Proposition 2.6.4.2 leads to the existence of an $(r+1)$-m.t.i.s. field, defined locally by means of the maps $\tilde{f}_{\alpha'}$.

2.6.6 Manifolds of Odd Dimension Admitting a Conformal Spin Special Structure

Let us assume that V is an orientable manifold of dimension $2r+1$. We extend the definitions given above, replacing respectively $RO(p+1,q+1)$, $C_n(p,q)$, and $PO(p+1,q+1)$ by $RO^+(p+1,q+1)$, $(C_n(p,q))_r$, and $PSO(p+1,q+1)$.

Cl^+_{n+2} is central, simple. $Cl^+_{n+2}(Q')$ ($n = 2r+1$, Q' the complexification of Q) is isomorphic to $Cl_{n+2}(Q')$ ($n = 2r$). We introduce the associated Witt basis

$\{x_i, y_j, z_n\}$ and associated projective Witt frame and the representation of Cl'^{+}_{n+2} in the space $x_{i_0} x_{i_1} \cdots x_{i_h} f_{r+1}$, $f_{r+1} = y_1 y_2 \cdots y_r y_0$. The bundles S_1 and σ_1 are defined in the same way. In the study of necessary and sufficient existence conditions, only a few details are modified: one arrives at identical statements, the $g_{\alpha'\beta'}(x)$ belonging to $RO^+(p+1, q+1)$. (Let us now recall that e_N belongs to the center of Cl_{n+2} and that $e_N f_{r+1} = f_{r+1} e_N = (-i)^{r-p} f_{r+1}$.)

2.7 Links between Spin Structures and Conformal Spin Structures

Let us assume that $n = p+q = 2r$. We study here only the case of real conformal spin structures in a strict sense. $\boxed{G}$ stands for the identity component of the Lie group G.

2.7.1 First Links

In the same way as in 2.6.1, we introduce the "Greub extension" ξ_j of ξ, the j-extension of ξ, and ξ_i, the i-extension of ξ, and then $P\xi_1 = \xi_{\tilde{\lambda}}$, the $\tilde{\lambda}$-extension of ξ.

Clif $_2$ is the auxiliary bundle, the typical fiber of which is $Cl_2(1,1)$. *Clif* (V, Q) is the Clifford bundle of (V, Q). According to the classical isomorphism (see, for example, Chapter 1), which we denote by λ_1 from $Cl_n(p,q) \otimes Cl_2(1,1)$ onto $Cl_{n+2}(p+1, q+1)$, we still abusively denote by λ_1 the isomorphism from *Clif* $(V) \otimes$ *Clif* $_2$ onto *Clif* $_1(V)$ and from *Clif*$'_1(V) \otimes$ *Clif*$'_2$ onto *Clif*$'_1(V)$.

Since $\Theta(V)$ is a trivial bundle, let us recall that then there exists a $RO(1,1)$-spin structure on $\Theta(V)$. ψ denotes the "twisted projection" from $RO(Q)$ onto $O(Q)$. We shall use the following two statements.[90]

There exists an $RO(p,q)$-spin structure in a strict sense on V iff:

(i) There exists on V, modulo a factor ± 1, an isotropic r-vector field, pseudo–cross section in the bundle *Clif* (V); the complexified pseudo-riemannian bundle $\xi_{\mathbf{C}}$ admits local cross sections, over a trivialization open set $(U_{\alpha'})_{\alpha' \in A}$ with transition functions $\psi(g_{\alpha'\beta'})$, $g_{\alpha'\beta'} \in RO^+(p,q)$ such that if $x \in U_{\alpha'} \bigcap U_{\beta'} \neq \emptyset \to f_{\alpha'}(x)$ locally define the previous r-vector field, then $f_{\beta'}(x) = N(\hat{g}_{\alpha'\beta'}(x)) f_{\alpha'}(x)$; $f_{\beta'}(x) = g_{\alpha'\beta'}(x) f_{\alpha'}(x) g^{-1}_{\alpha'\beta'}(x)$, where

$$f_{\alpha'}(x) = \mu^x_{\alpha'}(f_r), \quad \hat{g}_{\alpha'\beta'}(x) = \mu^x_{\alpha'}(g_{\alpha'\beta'}(x)),$$

$f_r = y_1 \cdots y_r$; $\mu^x_{\alpha'}$ is an isomorphism well-defined[91] from Cl'_n onto $Cl'_n(x)$.

(ii) The structure group of the bundle ξ is reducible in $O'(n)$ to a real spinoriality group $\sigma(p,q)$ in a strict sense.

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[90] A. Crumeyrolle, *Fibrations Spinorielles et Twisteurs Généralisés*, op. cit.
[91] Idem.

2.7.2 Other Links

Let us denote by a $C_n(p,q)$ spin structure, respectively by an $RO(p,q)$ spin structure, a real conformal spin structure in a strict sense, respectively a real $RO(p,q)$ spin structure in a strict sense, on V.

In the same way, we agree to denote by an $RO(p+1,q+1)$ spin structure a real $RO(p+1,q+1)$ spin structure over the bundle ξ_j of orthonormal frames of the amplified tangent bundle $T_1(V)$. We want to prove the following statement:

2.7.2.1 Proposition (1) *If there exists an $RO(p,q)$ spin structure on V, then there exists an $RO(p+1,q+1)$ spin structure on ξ_j.*

(2) *If there exists an $RO(p+1,q+1)$ spin structure on ξ_j, then there exists a $C_n(p,q)$ spin structure on V.*

(3) *If there exists an $C_n(p,q)$ spin structure on V, if r and p are odd, then there exists an $RO(p+1,q+1)$ spin structure on ξ_j.*

Proof. (1) Let us assume that there exists an $RO(p,q)$ spin structure on V. Let $f_r = y_1 \cdots y_r$ be an isotropic r-vector. By assumption, there exists a pseudo–cross section in the bundle $Clif'(V)$; so we can naturally form a pseudo–cross section in the bundle $Clif'(V) \otimes Clif'_2$, determined locally by

$$x \to \mu^{x}_{\alpha'}(f_r) \otimes \mu^{2x}_{\alpha'}(y_0) = f_{\alpha'}(x) \otimes f^{2}_{\alpha'}(x) = \tilde{f}_{\alpha'}(x),$$

where $x \to f^2_{\alpha'}(x) = \mu^2_{\alpha'}(y_0)$ determines locally a cross section in the bundle $Clif'_2$, with obvious notation. Using a λ_1 isomorphism from $Clif'(V) \otimes Clif'_2$ onto $Clif'_1(V)$, we obtain a pseudo–cross section in the bundle $Clif'_1(V)$ determined locally by means of $x \in U_{\alpha'} \to \tilde{f}'_{\alpha'}(x) = \lambda_1(\tilde{f}'_{\alpha'}(x))$ that satisfies the required conditions for the existence of an $RO(p,q)$ spin structure on ξ_j.[92]

Moreover, we observe that the reduction of $O(p,q)$ to $\sigma(p,q)$ in $O'(n)$ and that of $O(1,1)$ to $\sigma(1,1)$ in $O'(2)$ imply the reduction of $O(p+1,q+1)$ to $\sigma(p+1,q+1)$ associated with $y_1, \ldots, y_r y_0$ in $O'(n+2)$.

(2) Let us assume that there exists an $RO(p+1,q+1)$ spin structure on V. We observe that $\eta = \tilde{h} \circ \psi$ is a projection from $RO(p+1,q+1)$ onto $PO(p+1,q+1)$ with kernel $\mathcal{A} = \{1, -1, e_N, -e_N\}$.

There exists a principal bundle S_1 twofold covering of ξ_j and a morphism of principal bundles $\psi' : S_1 \to \xi_j$. So we can set $\tilde{\eta} = h \circ \psi'$, which is a morphism of principal bundles from S_1 onto $P\xi_1$, and S_1 is a fourfold covering of $P\xi_1$. Thus, we have obtained the existence of a $C_n(p,q)$ spin structure on V. We can also observe[93] that the reduction of $O(p+1,q+1)$ to $\sigma(p+1,q+1)$ in $O'(n+2)$ implies the reduction in $PO'(n+2)$ of $PO(p+1,q+1)$ to $\tilde{h}(\sigma(p+1,q+1))$, which is isomorphic by h_1 to $S_C(p,q)$, the real conformal spinoriality group in a strict sense associated with $f_r = y_1, \ldots, y_r$.

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[92] P. Anglès, Les structures spinorielles conformes réelles, Thesis, op. cit.

[93] Idem.

(3) Finally, let us assume that there exists a $C_n(p,q)$ spin structure on V and that r and p are odd. If r is odd, then $\varepsilon_2 = 1$. According to 2.6.3 and 2.6.4.1 above, there exists an isotropic $(r+1)$-pseudo–vector field (so defined modulo $\varepsilon_2 = 1$), locally determined by means of $x \in U_{\alpha'} \to \tilde{f}_{\alpha'}(x)$, such that for any $x \in U_{\alpha'} \bigcap U_{\beta'} \neq \emptyset$ we have

$$\tilde{f}_{\alpha'}(x) = \tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)\hat{g}^{-1}_{\alpha'\beta'}(x)$$

and

$$\tilde{f}_{\beta'}(x) = (e_N)^2 N(\tilde{g}_{\alpha'\beta'}(x))\tilde{f}_{\alpha'}(x) \text{ modulo } \varepsilon_2 = 1.$$

Thus, since r is odd, since $(e_N)^2 = (-1)^{r-p}$ if p is odd, then $(e_N)^2 = 1$. So we get the existence of an isotropic pseudo–vector field that satisfies the required sufficient condition[94] for the existence of an $RO(p+1, q+1)$ spin structure on ξ_j.

2.7.2.2 Remark Let us recall (2.5.1.6) that

$$\frac{\boxed{S_C(p,q)}}{\boxed{\sigma(p,q)}}$$

is homeomorphic to $\mathbf{R}^{2n+1}$ and so is a solid space.[95] Following corollary 12–6 in Steenrod,[96] any bundle with structure group $\boxed{S_C(p,q)}$ is reducible in $\boxed{S_C(p,q)}$ to a bundle with structure group $\boxed{\sigma(p,q)}$. If there exists a $C_n(p,q)$ spin structure on V, according to 2.6.4.1 $\boxed{C_n(p,q)}$ is reducible to $\boxed{S_C(p,q)}$ in C'_n.

Moreover, the previous reduction of $\boxed{S_C(p,q)}$ to $\boxed{\sigma(p,q)}$ is made in $\boxed{S_C(p,q)}$ and not in $O'(n)$, since $S_C(p,q)$ is obviously "extended out" of $O'(n)$, so that it is not permissible to use the sufficient condition given in 2.7.1 for the existence of an $RO(p,q)$ spin structure on V.

2.8 Connections: A Review of General Results[97]

2.8.1 General Definitions

Let $\xi = (P, \pi, M, G)$ be a differentiable principal fiber bundle. (For the sake of convenience we always assume that differentiability means that of class C^∞.) The total space P and the base M are differential manifolds and the projection π is a

[94] A. Crumeyrolle, *Fibrations Spinorielles et Twisteurs Généralisés*, op. cit.

[95] N. E. Steenrod, *The Topology of Fiber Bundles*, op. cit., p. 54. "We recall that a space Y will be called solid if for any normal space X, closed subset A of X and map $f : A \to Y$ there exists a map $f' : X \to Y$ such that $f' \mid_A = f$." (Cf. below footnote 126).

[96] Idem, p. 56.

[97] Cf., for example, S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 1, Interscience Publishers, New York, 1963; or Dale Husemoller, *Fiber Bundles*, 3rd edition, Springer-Verlag, 1994. Note that some authors use the term fiber for fiber.

differentiable mapping. The structure group G is a Lie group and acts on P on the right as a transformation group. On each fiber, G acts transitively without fixed points. For elements a, x in G, P, we write $R_a(x) = xa$.

Let p be an element in P such that $\pi(p) = b$ and let $T_p(P)$ be the tangent space at p and let $V_p = \ker(d\pi)_p$, where $(d\pi)_p$ is the tangent mapping of π in p. V_p is the subspace of $T_p(P)$ tangent the fiber $\pi^{-1}(b)$ at p. Elements of V_p are called vertical elements.

2.8.1.1 Definition A connection Γ in P is an assignment for each x in P of a subspace Q_x of $T_x(P)$ such that the following conditions are satisfied:

(i) $\pi_x(P) = V_x(P) \oplus Q_x$ (direct sum);

(ii) for every g in G and every x in P, $Q_{R_g(x)} = (dR_g)_x(Q_x)$ (i.e., the "distribution" $u \to Q_u$ is equivariant under G);

(iii) the mapping $x \to Q_x$ is differentiable (Q_x is called the horizontal subspace of $T_x(P)$).[98]

Let Γ be a connection in P. We define a 1-form w on P with values in the Lie algebra $\mathcal{G}$ of G as follows.

It is known that every $A \in g$ induces a vector field A^* on P, called the fundamental vector field corresponding to A, and that $A \to (A^*)_p$ is a linear isomorphism of $\mathcal{G}$ onto V_p, for each $p \in P$.[99]

2.8.1.2 Definition For each $X \in T_p(P)$ we define $w(X)$ to be the unique $A \in \mathcal{G}$ such that $(A^*)_p$ is equal to the vertical component of X. Thus, $w(X) = 0$ if and only if X is horizontal. The form w is called the connection form of the given connection Γ.

2.8.1.3 Proposition (Definitions[100]) *The connection form w of a connection satisfies the following conditions:*

(i)′ $w(A^*) = A$, *for every* $A \in \mathcal{G}$.

[98] A vector $X \in T_x(P)$ is called vertical, respectively horizontal, if it lies in V_x, resp. Q_x. According to (i), every vector $X \in T_x(P)$ can be uniquely written as $X = Y + Z$, where $Y \in V_x$ and $Z \in Q_x$. Y, resp. Z, is called the vertical, resp. the horizontal, component of X and denoted by $V(X)$, resp. $h(X)$. (iii) means that if X is a differentiable vector field on P, so are $V(X)$ and $h(X)$.

[99] For each A in $\mathcal{G}$, the 1-parameter subgroup $\exp t A$ $(-\infty < t < +\infty)$ defines a one-parameter group $R_{\exp tA}$ of transformations on P and it determines A^*, namely,

$$(A^* f)_p = \lim_{t \to 0} \frac{f(p.\exp tA) - f(p)}{t} = \frac{d}{dt} f(p.\exp tA)_{t=0}$$

for every p in P.

[100] Cf. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 1, op. cit., pp. 63–66.

(ii)′ $(R_g)^*w = ad(g^{-1}).w$, *i.e.*, $w((dR_g)X) = ad(g^{-1}).w(X)$ *for every* $g \in G$ *and every vector field* X *on* P, *where* ad *denotes the adjoint representation of* G *in* $\mathcal{G}$.

Conversely, given a g*-valued 1-form* w *on* P *satisfying conditions (i)′ and* $(ii)''$, *there is a unique connection* Γ *in* P *whose connection form is* w.

The projection $\pi : P \to B$ induces a linear mapping $\tilde{\pi} = d\pi : T_p(P) \to T_b(B)$ for each $p \in P$, where $b = \pi(p)$. When a connection is given, $\tilde{\pi}$ maps the horizontal subspace Q_p isomorphically onto $T_b(M)$.

The lift (or horizontal lift) of a vector field X on B is a unique vector field $\bar{X}$ on P that is horizontal and that projects onto X. The lift $\bar{X}$ is invariant by R_g, for every $g \in G$. Conversely, every horizontal field $\bar{X}$ on P invariant by G is the lift of a vector field X on M.

Let $(U_\alpha)_{\alpha \in A}$ be an open covering of M with a family of isomorphisms $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ and the corresponding family of transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. For each $\alpha \in A$, let $\sigma_\alpha : U_\alpha \to P$ be the cross section defined by $\sigma_\alpha = \varphi_\alpha^{-1}(x, e)$, $x \in U_\alpha$, where e is the identity of G.

The transition functions $g_{\alpha\beta}$ satisfy the consistency relations $g_{\alpha\beta}(x) g_{\beta\gamma}(x) = g_{\alpha\gamma}(x)$, for $x \in U_\alpha \bigcap U_\beta \bigcap U_\gamma$ and $\sigma_\beta(x) = \sigma_\alpha(x) g_{\alpha\beta}(x)$ for x in $U_\alpha \bigcap U_\beta$. We can define the pullback $w_\alpha = \sigma_\alpha^*(w)$ by σ_α of the 1-form w restricted to $\pi^{-1}(U_\alpha)$, which is a g-valued 1-form defined on U_α. We have the following classical statement.

2.8.1.4 Proposition *There exists a connection* Γ *with a* $Lie(G)$*-valued 1-form* w *on the principal fiber bundle if and only if for any* α, β *in* A,

$$w_\beta = ad(g_{\alpha\beta}^{-1}) w_\alpha + g_{\alpha\beta}^{-1} d g_{\alpha\beta}.$$

2.8.2 Parallelism

Given a connection Γ in a principal fiber bundle $\xi = (P, \pi, M, G)$, the following results concern the concept of parallel displacement of fibers along any curve γ in the base manifold M.

2.8.2.1 Definition Let $\gamma : t \to \gamma(t)$, $a \le t \le b$, be a piecewise differentiable curve of class C^1 in M; a lift (or horizontal lift) of γ is a horizontal curve $\varphi : t \to \varphi(t)$, $a \le t \le b$, such that $\pi \circ \varphi = \gamma$. Here, a horizontal curve in P means a piecewise differentiable curve of class C^1 whose tangent vectors are all horizontal.

Note: In what follows we will sometimes use the terms *curve* or *path* to *denote a differentiable curve of class* C^1.

2.8.2.2 Proposition (Definition) *Let* $\gamma : t \to \gamma(t)$, $0 \le t \le 1$, *be a curve of class* C^1 *in* M. *For an arbitrary point* p_0 *of* P *with* $\pi(p_0) = \gamma(0) = b_0$, *there exists a unique lift* $\varphi : t \to \varphi(t)$ *of* γ *that starts from* p_0.

We can now define the parallel displacement of fibers as follows: Let $\gamma : t \to \gamma(t)$, $0 \leq t \leq 1$, *be a differentiable curve of class* C^1 *on* M. *Let* p_0 *be an arbitrary point of* P *with* $\pi(p_0) = \gamma(0) = b_0$. *The unique lift* $\varphi : t \to \varphi(t)$ *of* γ *through* p_0 *has the endpoint* p_1 *such that* $\pi(p_1) = b_1 = \gamma(1)$.

By varying p_0 *in the fiber* $\pi^{-1}(b_0)$, *we obtain a mapping of the fiber* $\pi^{-1}(b_0)$ *onto the fiber* $\pi^{-1}(b_1)$, *which maps* p_0 *into* p_1. *We denote this mapping by* τ_γ *and call it the parallel displacement along the curve* γ. *The fact that* $\tau_\gamma : P_{b_0} = \pi^{-1}(b_0) \to P_{b_1} = \pi^{-1}(b_1)$ *is an isomorphism comes from the following proposition.*

2.8.2.3 Proposition *The parallel displacement along any curve* τ_γ *commutes with the action of* G *on* $P : \tau_\gamma \circ R_a = R_a \circ \tau_\gamma$, *for every* $a \in G$.

2.8.3 Curvature Form and Structure Equation

(Cf. exercises below)

2.8.3.1 Definition We call the curvature form Ω of the connection the 2-form on P with values in $\mathcal{G}$ defined by $\Omega(X, Y) = dw(h(X), h(Y))$, where $h(X)$ and $h(Y)$ are respectively the horizontal components of the vector fields X, respectively Y, defined on P.[101]

2.8.3.2 Theorem *Let* w *be a connection form and* Ω *its a curvature form. Then*

$$\Omega(X, Y) = dw(X, Y) + \frac{1}{2}[w(X), w(Y)]$$

for $X, Y \in T_p(P)$, $p \in P$ *(structure equation of Elie Cartan), which is sometimes written, for the sake of simplicity,*

$$\Omega = dw + \frac{1}{2}[w, w].$$

2.8.3.3 Theorem (Bianchi's identity) $D\Omega = 0$, *where* D *is the classical exterior covariant differentiation.*

2.8.3.4 Definition A connection Γ is called flat if its curvature form vanishes identically.

2.8.3.5 Theorem Ω *is equal to zero if and only if the field of horizontal subspace* $p \to Q_p$ *is involutive, i.e., if* X *and* Y *are two horizontal vector fields on* P, *then* $[X, Y]$ *is a horizontal vector field.*

101 Some authors, such as R. Deheuvels, *Tenseurs et Spineurs*, P.U.F., Paris 1993, chapitre IX-6, define Ω as follows: $\Omega(X, Y) = dw(X, Y) + [w(X), w(Y)]$, for every $X, Y \in T_p(P)$. Then Ω satisfies the following structure equation: $\Omega = dw + [w, w]$ and we have $d\Omega(h(X), h(Y), h(Z)) = 0$ for any $X, Y, Z \in T_p(P)$.

2.8.4 Extensions and Restrictions of Connections

2.8.4.1 Definitions

Let $P = (P, \pi, M, G)$ be a principal fiber bundle and let us assume that P has a reduced fiber bundle P'. We want to study the relation between the connections of P and of P'. Let H be a Lie subgroup of G and let $\mathcal{H}$ be its Lie algebra. We will denote by j both the injection of H into G and the injection of $\mathcal{H}$ into $\mathcal{G}$. If there exist a differentiable principal fiber bundle $P' = (P', \pi', M, H)$ and a differentiable embedding f of P' into P such that $\pi \circ f = \pi'$ and $f \circ R_a = R_{j(a)} \circ f$, for every $a \in H$, are satisfied, then (P', f) is said to be a reduced fiber bundle of P. Then we have $df(A_x^*) = j(A)^*_{f(x)}$, for every $A \in \mathcal{H}$ and $x \in P'$ (with the previous definition of A^* (cf. 2.8.1)).

Given a connection in P', we denote the horizontal space at the point x of P' by Q'_x. At the point $f(x)$ of P, we take $df(Q'_x)$ as the horizontal space and transform it by right translations of G. Thus we obtain a connection on P. Let us denote respectively by w and w' the corresponding connection forms. Then we have $j \circ w' = f^*(w)$ on P (here, for a mapping l, l^* denotes the "pull back" of l).

Conversely, let us assume that a connection is given in P with the connection form w. If the induced form $f^*(w)$ on P' has values always in $j(\mathcal{H})$, we can write $f^*(w) = j \circ w'$, and w' defines a connection in P'. Thus, the connection in P is called an extension of the connection in P', and the connection in P' is called the restriction of the connection in P.

2.8.4.2 Connections Associated with a Principal Connection[102]

2.8.4.2.1 Definitions

Let $\xi = (P, \pi, B, G)$ be a principal bundle. Let R_g be the right action on P of $g \in G$. We write $R_g(p) = p \cdot g$, $p \in P$. Let F be a C^∞-differentiable manifold. We assume that G acts differentiably on the left on F. We denote by L_g the left action on F of $g \in G$ and we write $L_g(f) = g \cdot f$, $f \in F$. We define a right action of G on $P \times F$ by assuming that

$$(p, f) \cdot g = (p \cdot g, g^{-1} \cdot f), \quad p \in P, \ f \in F, \ g \in G.$$

The quotient space $E = P \times_G F$ inherits a fibered structure with base B, fiber F. Such a bundle is said to be associated with P. Let π_E be the canonical projection from E onto B. Any point p in P defines a diffeomorphism, denoted by $\tilde{p}$, of F on $\pi_E^{-1}(b)$, where $b = \pi(p)$: we associate with any $f \in F$ the class $\overline{(p, f)}$, which we agree to denote again by $p \cdot f$, constituted by the elements $(p \cdot g, g^{-1} \cdot f)$ of $P \times F$.

2.8.4.2.2 Example Let V be a C^∞ differentiable manifold of dimension n. Let $R(V)$ be the principal bundle of frames of V. The fiber bundle with typical fiber $\mathbf{R}^n$

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102 Cf., for example, Phan Mau Quan, *Introduction à la Géométrie des Varietés Différentiables*, Dunod, Paris, 1969.

associated with $R(V)$ can be identified with the tangent fiber bundle of the manifold $T(V)$. For any $p \in R(V)$, $\tilde{p}$ is then the diffeomorphism from $\mathbf{R}^n$ onto $T_{\pi(p)}(V)$ defined by the frame P.

2.8.4.2.3 Theorem *Any principal connection on ξ defines a unique differential system $K : z \to K_z$ on E such that for any point z in E, K_z is a complementary subspace in $T_z(E)$ of W_z, the subspace of vertical vectors in z.*

Let $z_0 \in E$ and $p_0 \in \pi^{-1}(\pi_E(z_0))$. There exists a unique $f_0 \in F$ such that $\tilde{p}_0(f_0) = z_0$. We put $K_{z_0} = (d\phi_{f_0})_{p_0}(H_{p_0})$ where ϕ_{f_0} is the mapping from P into E defined by $\phi_{f_0}(p) = \tilde{p}(f_0) = p \cdot f_0$. K_{z_0} is independent of the choice of $p_0 \in \pi^{-1}(\pi_E(z_0))$, according to the invariance of H_{p_0} for the action of G on P and satisfies the relation $T_{z_0}(E) = W_{z_0} \oplus K_{z_0}$.

2.8.4.2.4 Definition K is called the connection associated with the principal connection H.

A continuously differentiable path $\psi : t \to z_t = \psi(t)$ in E is called horizontal if $\dot{z}_t \in K_{z_t}$, for any t. (Cf. footnote 110.) As previously, one defines the notion of horizontal lift of a path $\gamma : t \to b_t = \gamma(t)$ of B. We can give the following result:

2.8.4.2.5 Theorem *If $\gamma : t \to b_t = \gamma_t$ is a continuously differentiable path in B, for any z_0 in the fiber $\pi_E^{-1}(b_0)$ over b_0, there exists a unique horizontal lift of γ, with origin z_0.*

Let p_0 be in P and f_0 in F such that $\tilde{p}_0(f_0) = z_0$. We define the lift by considering the path $t \to z_t$ of E, where $z_t = \tilde{p}_t(f_0)$, the path $t \to p_t$ of P being the unique horizontal lift of γ in P, with origin p_0. Let γ be a continuously differentiable path in B with origin b_0 and endpoint b_1. Let E_{b_0}, respectively E_{b_1}, be the fibers over b_0, respectively b_1. We can associate with any point z_0 in E_{b_0} a unique horizontal lift of γ with origin z_0. Let z_1 be its endpoint in E_{b_1}. We define, therefore, a mapping $\tau_\gamma^E : E_{b_0} \to E_{b_1}$.

2.8.4.2.6 Definition τ_γ^E is a diffeomorphism from E_{b_0} onto E_{b_1} called the parallel displacement in E corresponding to the path γ.

2.8.4.2.7 Remark Let (z_t) be a horizontal lift of the continuously differentiable path γ. We have $z_t = \tilde{p}_t(f_0)$, where (p_t) is a horizontal lift of (b_t) in P and f_0 the unique element in F such that $z_0 = \tilde{p}_0(f_0)$, according to Theorem 2.8.4.2.5.

Therefore, we deduce that $z_t = \tilde{p}_t \circ \tilde{p}_0^{-1}(z_0)$, and then $\tau_\gamma^E = \tilde{p}_t \circ \tilde{p}_0^{-1}$.

2.8.5 Cartan Connections[103]

2.8.5.1 Classical Definitions

Let M be a differentiable manifold of dimension n. Consider a homogeneous space $F = G/H$ of the same dimension n, where G is a Lie group and H is a closed subgroup of G. Let $B = (B, M, F, G)$ be a fiber bundle over M with fiber F and structure group G and let $P = (P, M, G)$ be the principal fiber bundle associated with B.

Suppose that there exists a cross section f over M to B. Then the structure group of P can be reduced to H. We denote this reduced fiber bundle by $P' = (P', M, H)$ and the injection of P' into P by j.

2.8.5.2 Definition Let us assume that a connection Γ is given in P. Its connection form w is a differential form of degree 1 on P, with values in $Lie(G)$, and the induced form $w' = j^*(w)$ is also a differential form of degree 1 on P' with values in $Lie(G)$. We call the connection in P a Cartan connection on M with the fiber $F = G/H$ if at each point p' of P', $w'_{p'}$ gives an isomorphism of $T_{P'}(P')$ onto g as linear spaces.

We have the following equivalent definition:

2.8.5.3 Definition Let w' be a 1-form on P' with values in g satisfying the following three conditions:

(i) $w'(A^*) = A$, for every $A \in Lie(H)$, Lie algebra of H;

(ii) $R_a^*(w') = ad(a^{-1})w'$, for every $a \in H$;

(iii) $w'_{p'}$ gives an isomorphism of $T_{p'}(P')$ onto $Lie(G)$, at each point $p' \in P'$.

For such w', we can take a connection form w in P such that $w' = j^*(w)$. w defines a Cartan connection.

2.8.6 Soudures (Solderings)[104]

We use the same definitions as in 2.8.5.1.

A cross section f over M to B gives a vector bundle $T'(B)$ on M as follows: for every point p of M, the projection $B \to M$ defines a mapping $T_{f(p)}(B) \to T_p(M)$. The kernel of this mapping is denoted by $V_{f(p)}(B)$. Then $T'(B) = \cup_p V_{f(p)}(B)$ forms a vector bundle over M and the dimension of its fibers is equal to $n = \dim F$.

A Cartan connection in P gives a bundle isomorphism between $T'(B)$ and the tangent vector bundle $T(M)$ as follows. Let p' be an arbitrary point in P' and let

103 J. Dieudonné, *Elements d'Analyse*, Tome 4, Gauthier-Villars, 1971, p. 241, or S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer-Verlag, New York, 1978, pp. 127–130.

104 C. Ehresmann, Les connections infinitésimales dans un espace fibré différentiable, *Colloque de Topologie*, Brussels, 1950, pp. 29–55.

us put $q = \pi(p')$. *The projection* $\pi : P' \to M$ *induces an isomorphism of* $T_{p'}(P')/V_{p'}(P')$ *onto* $T_q(M)$. *On the other hand,* $w'_{p'}$ *gives an isomorphism of* $T_{p'}(P')/V_{p'}(P')$ *onto* $Lie(G)/Lie(H)$. *As a point in* P', p' *gives a mapping of* $F = G/H$ *onto the fiber in* B *over* q *and sends the point* $\{H\}$ *in* F *to* $f(q)$. *By this mapping* $T_0(F) = Lie(G)/Lie(H)$ *is mapped isomorphically onto* $V_{f(q)}(B)$. *Combining these isomorphisms, we obtain an isomorphism between* $T_q(M)$ *and* $V_{f(q)}(B)$ *that is independent of the choice of* $p' \in P'$ *over* q.

The set of such isomorphisms for $q \in M$ *defines a bundle isomorphism of* $T(M)$ *and* $T'(B)$. *If a fiber bundle* B *over* M *has an isomorphism such as above through a cross section, then* B *is said to have a soudure.*

Conversely, if a fiber bundle B *over* M *has a soudure with respect to a cross section* f, *then, there exists a Cartan connection in* P *such that the soudure given by the connection is the original one.*[105]

2.8.7 Ehresmann Connections

We want to present some specific results of Ehresmann.[106]

Let (M, π, N) be a differentiable locally trivial bundle, with $\dim N = n$, and $\dim M = m + n$. Let $V(M)$ be the vertical subbundle of $T(M)$ and $\pi^*(T(N))$ the induced bundle of $T(N)$ under π.[107]

The maps $(d\pi)_z : T_z(M) \to T_{\pi(z)}(N)$ lead to the morphism of vector bundles $\tilde{d\pi} : T(M) \to \pi^*(T(N))$, corresponding to the following exact sequence of vector bundles:

$$0 \longrightarrow V(M) \longrightarrow T(M) \xrightarrow{\tilde{d\pi}} \pi^*(T(N)) \longrightarrow 0.$$

2.8.7.1 Definition A morphism Γ of vector bundles from $\pi^*(T(N))$ into $T(M)$ is called a horizontal morphism of the bundle (M, π, N) if it satisfies the following condition:

$$\tilde{d\pi} \circ \Gamma = id_{\pi^*(T(N))}.$$

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[105] C. Ehresmann, Les connections infinitesimales dans un espace fibré différentiable, op. cit.

[106] R. Hermann, *Gauge Fields and Cartan–Ehresmann Connections*, Part A, Math. Sci. Press, Brookline, 1975; and L. Mangiarotti and M. Modugno, Graded Lie algebras and connections on a fibered space, *J. Maths. Pures et Appl.*, 63, 1984, pp. 111–120.

[107] *Some classical definitions:*

(a) Let $\xi = (E, p, B)$ and $\xi' = (E', p', B')$ two vector bundles, a morphism of vector bundles $(u, f) : \xi \to \xi'$ is a morphism of the underlying bundles, that is, $u : E \to E'$, $f : B \to B'$ are maps such that $p'u = fp$, and the restriction $u : p^{-1}(b) \to p'^{-1}(f(b))$ is linear for each $b \in B$.

(b) Let $u : \xi \to \eta$ be a morphism of vector bundles over B. We define $Im\, u$ to be the subbundle of η with total space the subspace of $E(\eta)$ consisting of all $u(x)$, $x \in E(\xi)$.

(c) Let $\xi = (E, p, B)$ be a bundle and let $f : B_1 \to B$ be a map. The induced bundle of ξ under f, denoted by $f^*(\xi)$, has as base space B_1, as total space E_1, which is the subspace of all $(b_1, x) \in B_1 \times E$ with $f(b_1) = p(x)$, and as projection p_1, the map $(b_1, x) \to b_1$ (cf., for example, Dale Husemoller, *Fiber Bundles*, 3rd edition, Springer-Verlag, 1994, chapter 2, pp. 11–37).

Let Γ be a horizontal morphism. Then we have $T_z(M) = \operatorname{Im}\Gamma_z \oplus V_z(M)$[106], for each z in M. Then $\operatorname{Im}\Gamma_z$ constitute the fibers of a horizontal subbundle $H(M)$ of $T(M)$, that is, of a vector subbundle of $T(M)$ such that $T(M) = H(M) \oplus V(M)$.

Conversely, given a horizontal subbundle $H(M)$, one can define a horizontal morphism of the bundle. Then the set of horizontal morphisms of the bundle is equipotent with the set of horizontal subbundles of $T(M)$.[108]

We have the following classical result.[109]

2.8.7.2 Proposition *There always exists a horizontal subbundle of $T(M)$.*

2.8.7.3 Local Characterization

Let Γ be the horizontal morphism corresponding to a horizontal subbundle $H(M)$ of $T(M)$. Let $z \in M$ and let U be an open set belonging to an atlas of the bundle (M, π, N) such that $\pi(z) \in U$. There exists a diffeomorphism ϕ from $\pi^{-1}(U)$ onto $U \times F$, where F is a typical fiber, such that $\pi \circ \phi(x, y)^{-1} = x$, $x \in U$, $y \in F$. We can assume that U is a domain of coordinates $(x^\lambda)_{\lambda=1,2,\ldots,n}$ for the manifold N. Let W be a domain of coordinates $(y^i)_{i=1,2,\ldots,m}$ for the manifold F. $\{x^\lambda, y^i\}$, with $\lambda = 1, 2, \ldots, n$, and $i = 1, 2, \ldots, m$, is a system of coordinates on the neighbourhood of z, $\phi^{-1}(U \times W)$.

Thus we obtain the following system of local coordinates of $T(M)$: $\{x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i\}$, with obvious notation. Let X be a vector field defined on U. X can be written $X = X^\lambda \partial_\lambda$, where $X^\lambda \in C^\infty(U)$ and $\partial_\lambda = \partial/\partial x^\lambda$ such that $X^\lambda = \dot{x}^\lambda(X)$.[110]

Since Γ is a morphism of vector bundles, we can write $\Gamma(X) = X^\lambda(A^\mu_\lambda \partial_\mu + A^i_\lambda \partial_i)$, where $\partial_i = \partial/\partial y^i$ and A^μ_λ and A^i_λ are C^∞ functions defined on M.

Moreover, Γ satisfies the condition $\tilde{d}\pi \circ \Gamma = id_{\pi^*(T(M))}$. Then we have $A^\mu_\lambda = \delta^\mu_\lambda$, and therefore we obtain the local following characterization: In the system of local coordinates $\{x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i\}$,

$$(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i) \circ \Gamma = (x^\lambda, y^i, \dot{x}^\lambda, -\Gamma^i_\lambda \dot{x}^\lambda),$$

(Γ^i_λ) functions in $C^\infty(M)$. Therefore, $H(M)$ is generated by the following vector fields: $\partial_\lambda - \Gamma^i_\lambda \partial_i$. A horizontal piecewise differentiable curve of class C^1 in $M : t \to p(t)$ that is such that $\dot{p}_t \in H_{p_t}$ for every t, locally defined by $x^\lambda(p(t)) = x^\lambda(t)$, $y^i(p_t) = y^i(t)$ satisfies the equation

$$\frac{dy^i}{dt} = -\Gamma^i_\lambda \frac{dx^\lambda}{dt}.$$

[108] Cf. Greub, Halperin, Vanstone, *Connections, Curvature and Cohomology*, vol. 2, Academic Press, 1972, chapter VII, sect. 6.

[109] Greub, Halperin, Vaustone, *Connections, Curvature and Cohomology*, vol. 1, Academic Press, 1972, Proposition VII, p. 68.

[110] Each dotted letter denotes the tangent vector at that point; that is, $\dot{u}_j$ is the vector tangent to the curve u_j at the point u_j.

2.8.7.4 Definition Let s be a cross section of the bundle (M, π, N). The local 1-form on N with values in the vector bundle $V(M)$ defined by $\nabla s = K_M \circ ds$, where K_M denotes the projection $T(M) \to V(M))$ associated with $H(M)$, is called the covariant derivative of s.

2.8.7.5 Local Characterization of ∇s

In the previous system of local coordinates, let s be a local cross section defined on the open set U and let $X = X^\mu \partial_\mu$ be a vector field defined on U; $ds(X)$ can be calculated locally. We put $s^i = y^i \circ s$:

$$\begin{aligned} ds(X^\mu \partial_\mu) &= X^\mu \partial_\mu + (\partial_\lambda s^i) X^\lambda (\partial_i \circ s) \\ &= X^\mu(\partial_\mu - \Gamma^i_\mu \circ s(\partial_i \circ s)) + X^\mu(\partial_\mu s^i + \Gamma^i_\mu \circ s)(\partial_i \circ s). \end{aligned}$$

Therefore, according to 2.8.7.3,

$$K_M \circ ds(X) = X^\mu(\partial_\mu s^i + \Gamma^i_\mu \circ s)(\partial_i \circ s),$$

whence

$$\nabla s = (\partial_\mu s^i + \Gamma^i_\mu \circ s) dx^\mu \otimes (\partial_i \circ s).$$

2.8.7.6 Definition The mapping $\Omega : T(N) \times T(N) \to V(M)$ defined by $\Omega(X, Y) = -K_M([\Gamma(X), \Gamma(Y)]) = \Gamma([X, Y]) - [\Gamma(X), \Gamma(Y)]$, $X, Y \in T(N)$, is called the curvature of the horizontal subbundle $H(M)$.

Ω is a $C^\infty(N)$-skew-symmetric bilinear mapping that satisfies the following proposition.

2.8.7.7 Proposition Ω *vanishes identically if and only if* $H(M)$ *is involutive.*

2.8.7.8 Remark Let us consider a connection with the form w on a principal bundle P with structure group G—defined by a horizontal subbundle of $T(P)$—whose curvature form (cf. Definition 2.8.3.1) satisfies the relation

$$\Omega(X, Y) = dw(h(X), h(Y)) = -w([h(X), h(Y)]).$$

We find that both definitions are equivalent, since for any p in P the mapping $A \to A^*_p$ from the Lie algebra $Lie(G)$ of G into the space V_p of vertical elements is an isomorphism of vector spaces.

2.8.7.9 Local Characterization

$\Omega = \Omega^i \partial_i$ with $\Omega^i = (\partial_\lambda \Gamma^i_\mu - \Gamma^j_\lambda(\partial_j \Gamma^i_\mu)) dx^\lambda \wedge dx^\mu$. Let us consider the local basis $\theta^i = dy^i + \Gamma^i_\mu dx^\mu$ in $P(H(M)) = \{\theta \in D_1(M) : \theta(X) = 0, \forall X \in H(M)\}$. We have

$$d\theta^i = \pi^*(\Omega^i) + (\partial_j \Gamma^i_\lambda)\theta^j \wedge dx^\lambda.$$

$D_1(M)$ denotes the module of the derivations of M. We recall that a derivation of an algebra A over a field K is a K–linear transformation of A such that for any $f, g \in A$ we have $D(fg) = D(f)g + fD(g)$. For a manifold M the set $\mathcal{F} = C^\infty(M)$ of differentiable functions on M constitutes an algebra over $\mathbf{R}$. Any derivation of $\mathcal{F}$ is called by definition a vector field over M. $D_1(M)$ denotes the set of such vector fields and $D_1(M)$ is a module over $\mathcal{F}$. The local expression can be obtained by computation of $d\theta^i$ and according to the following *theorem of Frobenius*.[111]

2.8.7.10 Theorem *The following conditions are equivalent:*

(i) *$H(M)$ is involutive.*
(ii) *For any z_0 in M, there exists a connected maximal unique submanifold N_{z_0} of M such that $z_0 \in N_{z_0}$ and $T_z(N_{z_0}) = H_z(M)$ for any z in N_{z_0}.*
(iii) *For any element θ in $P(H(M))$,*

$$d\theta = \sum_{i=1}^{m} \eta_i \wedge \theta^i,$$

where $\{\theta^i\}$ form a local basis of $P(H(M))$ and $\{\eta_i\}$ are m differential 1-forms on M.

2.8.7.11 Definition Let $H(M)$ be a horizontal subbundle. Let $t \to q(t)$ and $t \to p(t)$ be piecewise differentiable curves on N, respectively M. We say that $t \to p(t)$ is a horizontal lift of the curve $t \to q(t)$ if $\pi(p(t)) = q(t)$ and $\dot{p}(t) \in H_{p(t)}(M)$ for any t.

2.8.7.12 Definition A horizontal subbundle $H(M)$ of $T(M)$ is called an Ehresmann connection if it satisfies the following condition:

For any piecewise differentiable curve q_t, $t_0 \leq t \leq t_1$, defined on N and for any $p_0 \in \pi^{-1}(q(t_0))$, there exists a unique horizontal lift (p_t), $t_0 \leq t \leq t_1$ such that $p_{t_0} = p_0$.

Let γ a piecewise differentiable curve on N that starts from q_0 and ends at q_1. Let M_{q_0} and M_{q_1} be the fibers over q_0, respectively q_1. For any point p_0 of M_{q_0}, there exists a unique horizontal lift of γ that starts from p_0. Let p_1 be its endpoint in M_{q_1}. We can define the parallel displacement of fibers τ_γ from M_{q_0} into M_{q_1} along the curve γ.

2.8.7.13 Definition τ_γ is a diffeomorphism from M_{q_0} onto M_{q_1} called the parallel displacement along the curve γ.

2.8.7.14 Example (i) Any principal connection on a principal bundle P is an Ehresmann connection.

[111] Cf., for example, S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 1, op. cit., p. 10, Proposition 1.2. Phan Mau Quan, *Introduction à la Géométrie des Variétés Différentiables*, Dunod, Paris, 1968, p. 102. Y. Choquet-Brubet, *Géométrie Differentielle et Systèmes Extérieurs*, Dunod, Paris, 1968, p. 192 and p. 197.

(ii) According to 2.8.4.2.3 and 2.8.4.2.5, any connection associated with a principal connection is an Ehresmann connection. But an Ehresmann connection on a bundle E associated with a principal bundle P is not always associated with a principal connection on P. If the structure group of the bundle P reduces to $\{e\}$, there exists a unique principal connection on P—the trivial one—while there can exist many Ehresmann connections on E (cf. below, exercises).

2.8.8 Ehresmann Connection in a Differentiable Bundle with Structure Group G, a Lie Group[112]

2.8.8.1 Definition A differentiable—locally trivialized—bundle, with structure group G is a fiber bundle (M, π, N) with typical fiber F such that:

(i) G acts differentially and effectively on the left on F,

(ii) there exist a trivializing atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of the bundle and mappings $g_{\alpha\beta}$: $U_\alpha \cap U_\beta \to G$ such that for any x in $U_\alpha \cap U_\beta : \varphi_{\alpha\beta}(x) = L_{g_{\alpha\beta}(x)}$, where $\varphi_{\alpha\beta}$ denote the transition functions of the bundle for the atlas $(U_\alpha, \varphi_\alpha)$ and $L_{g_{\alpha\beta}(x)}$ the corresponding diffeomorphisms from F onto F induced by the left action of the $g_{\alpha\beta}(x)$.

Such an atlas is called a G-trivializing atlas.

Let (M, π, N, F, G) be a bundle with structure group G, a Lie group, and $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ a G-trivializing atlas. For any x in N, let P_x be the set of diffeomorphisms h_x from F onto the fiber at x, M_x such that if h_x and l_x both are elements of P_x, then $h_x^{-1} \circ l_x$ is the diffeomorphism induced by the action on F of g in G. Let $\mathbf{P}$ be

$$\bigcup_{\mathbf{x} \in \mathbf{N}} \mathbf{P_x}.$$

The mapping $\pi_P : P \to N$ defined by $\pi_p(h_x) = x$ is a surjective mapping. G acts on the right on P by $h_x \circ g = h_x \circ L_g$. Since G acts effectively on F, such an action is a free one. The mappings $\psi_\alpha : (x, g) \in (U_\alpha \times G) \to \varphi_{\alpha,x} \circ L_x \in \pi_p^{-1}(U_\alpha)$ are bijective mappings. The mappings $\psi_{\alpha\beta} : (x, g) \in (U_\alpha \cap U_\beta) \times G \to (\psi_\alpha^{-1} \circ \psi_\beta)(x, g) \in (U_\alpha \cap U_\beta) \times G$ are diffeomorphisms. From this we deduce the following theorem:

2.8.8.2 Proposition (Definition) *P is a principal bundle with base space N, projection π_P, structure group G. P is called the principal bundle associated with (M, π, N, F, G).*

There exists the structure of a manifold on P for which P is a differentiable bundle.[113]

One can verify that P is a principal bundle and that $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ is a trivializing atlas for this bundle with cocycles the $g_{\alpha\beta}$.

[112] C. Ehresmann, Les connections infinitésimales dans un espace fibré différentiable, *Colloque de Topologie*, Brussels, 1950, pp. 29–55.

[113] Greub, Halperin, Vanstone, *Connections, Curvature and Cohomology*, vol. 1, Academie Press, Proposition X, p. 39.

2.8.8.3 Definition An Ehresmann connection $H(M)$ on a bundle with structure group G, (M, π, N, F, G) is called a G-Ehresmann connection if it satisfies the following condition: for any piecewise differential curve γ of N, the parallel displacement τ_γ along γ, viewed as a diffeomorphism from F onto F (by identification of fibers and typical fibers) is the diffeomorphism induced by the left action on F of an element in G.

2.8.8.4 Proposition *Let (M, π, N, F) be a bundle with structure group G, a Lie group, and let P be its corresponding principal bundle. G-Ehresmann connections on (M, π, N, F) are connections associated with principal connections on P.*

The proof will be given in the exercises.

2.8.8.5 Proposition *Let $H(M)$ be a G-Ehresmann connection on the bundle (M, π, N, F, G). According to Proposition 2.8.8.4, $H(M)$ is associated with a principal connection with form w on the principal bundle P associated with (M, π, N, F, G).*

Let $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ be a G-trivializing atlas for the bundle (M, π, N, F, G). Let $t \to z_t$ be a horizontal piecewise differential curve in $\pi^{-1}(U_\alpha)$ and (ψ_t) the corresponding piecewise differential curve in P defined by $\psi_t = \sigma_\alpha(\gamma_t)$, where σ_α is the local cross section in P over U_α defined by $\sigma_\alpha(x) = \varphi_{\alpha,x}$, $x \in U_\alpha$, and γ_t the projection onto N of the curve z_t. The piecewise differential curve (y_t) in F defined by $y_t = \psi_t^{-1}(z_t)$ satisfies the following equation: $\dot{y}_{t_0} = (\mu(-(\sigma_\alpha^*.w)(\dot{\gamma}_{t_0})))_{y_{t_0}}$, where μ is the isomorphism of Lie algebras from g onto $D^1(F)$[114] defined by

$$(\mu(A))_y = \frac{d}{dt}(L_{\exp(tA)} \cdot y)_{t=0}, \quad A \in g,\ y \in F.$$

The proof will be given below in the exercises.

We can now give the relation between the connection form w on the associated principal bundle and local forms of the Ehresmann connection defined by $\phi^i = \Gamma^i_\lambda dx^\lambda$.

2.8.8.6 Characterization

Let (E_I) be a basis for the Lie algebra Lie(G). The isomorphism μ from Lie(G) onto $D^1(F)$ is defined by $\mu(E_I) = \mu^i_I \partial_i$, $\mu^i_I \in C^\infty(F)$. If we set $w = w^I E_I$, with w^I 1-forms defined on P, we deduce from 2.8.8.5 and 2.8.7.3 that

$$\phi^i = \mu^i_I (\sigma_\alpha \circ \pi)^* w^I.$$

We can deduce the relation between the curvature form Ω on the associated principal bundle P and the local forms of the curvature of the Ehresmann connection $H(M)$.

2.8.8.7 Proposition *The local components (Ω^i) of the curvature Ω of the Ehresmann connection $H(M)$ are*

$$\Omega^i = \mu^i_I \sigma_\alpha^* . \Omega^I.$$

[114] Definition: $D^1(F)$ denotes the space of vector fields on F.

2.8.8.8 Example Let F be a real m-dimensional vector space and let $\{e_i\}_{i=1,\dots,m}$ be a given basis of F. We consider a differentiable—locally trivialized—bundle (M, π, N, F) with structure group $GL(F)$—the group of linear isomorphisms from F onto F. We can provide this bundle with the structure of a vector bundle by using diffeomorphisms $\varphi_{\alpha,x}$ associated with a $GL(F)$-trivializing atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$, for the transfer of the structure of vector space of F on the fibers. Let $H(M)$ be a $GL(F)$-Ehresmann connection on the vector-bundle (M, π, N, F).

According to Proposition 2.8.8.4, $H(M)$ is associated with a connection with 1-form w with values in the Lie algebra $gl(F)$ of $GL(F)$ on the associated principal bundle P, called the bundle of frames.

The local expression will be studied below in the exercises.

2.8.8.9 Example Let F be an affine space with dimension m and let $\{0, \{e_i\}\}$ be a given frame in F. We consider a differentiable bundle (M, π, N, F) with structure group the Lie group $A(F)$ of affine transformations of F.

2.8.8.10 Definition We call any $A(F)$-Ehresmann connection on the bundle (M, π, N, F) an Ehresmann affine connection.

According to Proposition 2.8.8.4, $H(M)$ is associated with a connection with form $\tilde{w}$ with values in the Lie algebra $a(F)$ of $A(F)$ on the principal bundle P associated with (M, π, N, F). $A(F)$ is isomorphic to the semidirect product of $GL(F)$ by F, since any affine transformation on F is the composite of a linear isomorphism on the vector space associated with F and a translation. The Lie algebra $a(F)$ can be written as $a(F) = gl(F) \oplus F$. The local expression of $H(M)$, the study of the curvature $\tilde{\Omega}$ of the Ehresmann affine connection $H(M)$, and the specific study of the Ehresmann affine connection on a vector bundle will be given below in the exercises.

2.9 Conformal Ehresmann and Conformal Cartan Connections

2.9.1 Conformal Ehresmann Connections[115]

Let M_n be the Möbius space (cf. 2.4.1) associated with the standard pseudo-Euclidean space $E_n(p, q)$. Let $C_n(p, q)$ be the conformal group of $E_n(p, q)$ viewed as the restriction of $PO(p+1, q+1) = \frac{O(p+1,q+1)}{\mathbf{Z}_2}$ to the Möbius space M_n. Since $PO(p+1, q+1)$ acts on the projective space $P(E_{n+2})$, we can give two definitions.

2.9.1.1 Definition A conformal Ehresmann connection is a G-Ehresmann connection with $G = PO(p+1, q+1)$ on a fiber bundle ξ with typical fiber M_n and structure group $PO(p+1, q+1)$.

[115] Most of the results given in Section 2.9.1 can be found in J. L. Milhorat, Sur les connections conformes, Thesis, Université Paul Sabatier, Toulouse, 1985.

2.9.1.2 Definition A projective Ehresmann connection is a G-Ehresmann connection with $G = PO(p+1, q+1)$ on a fiber bundle η_1 with typical fiber $P(E_{n+2})$ and structure group $PO(p+1, q+1)$.

We want to study some properties of conformal Ehresmann connections and the links between conformal Ehresmann connections according to Definition 2.9.1.1 and projective Ehresmann connections according to Definition 2.9.1.2.

2.9.1.3 Local Characterization

In 2.4.2 we have defined u as the injective mapping from $E_n(p,q)$ into the isotropic cone of $E_{n+2}(p+1, q+1)$ by $u(x) = x^2 x_0 + x - y_0 = q(x)x_0 + x - y_0$ with $x_0 = (e_0 + e_{n+1})/2$ and $y_0 = (e_0 - e_{n+1})/2$.

For the sake of convenience we put now, once and for all, $y_{n+1} = (e_0 - e_{n+1})/2$. *We recall that* $2B(x_0, y_{n+1}) = 1$.

2.9.1.3.1 Lemma *Let* $z = \alpha x_0 + x + \beta y_{n+1}$ *be an element of* $E_{n+2}(p+1, q+1)$, *with* $x \in E_n(p,q)$ *and* $(\alpha, \beta) \in \mathbf{R}^2$. *Any element of* M_n, *the Möbius space, is the class* $\bar{z}$ *of an element* $z = \alpha x_0 + x + \beta y_{n+1}$ *that satisfies the condition* $\alpha\beta + \mathbf{q(x)} = \mathbf{0}$ *as* $M_n = P(Q(F)\backslash\{0\})$ *with previous notation (1.4.3.2; 2.4.1), where* P *is the canonical projection from* $E_{n+2} = F$ *onto its projective space and* Q *denotes the isotropic cone of* F.

We define the open set $\mathcal{U}$ of M_n by the set of $\bar{z}$ with $\beta \neq 0$ and $\bar{z} = \overline{(u(-x/\beta))} = \overline{(-(\alpha/\beta)x_0 - (x/\beta) - y_{n+1})}$ as $\alpha\beta = -q(x)$ and $q(-x/\beta) = (1/\beta^2)q(x)$. One can verify immediately that the mapping $\hat{\varphi}$, $\bar{z} \to -x/\beta$ is a homeomorphism from $\mathcal{U}$ onto $E_n(p,q)$.

Thus, $(\mathcal{U}, \hat{\varphi})$ *is a local chart of* M_n.

2.9.1.3.2 Local Characterization

Let $\xi = (\bar{M}, \bar{\pi}, N, M_n, PO(p+1, q+1))$ *be a bundle with typical fiber* M_n *and structure group* $PO(p+1, q+1)$. *We put now* $\dim N = m$ *and we recall that* $\dim M_n = n$ *and that* $\dim \bar{M} = n + m$.

According to 2.8.8.4, $H(\bar{M})$ is associated with a principal connection with form $\bar{w}$ on the principal bundle P_ξ associated with ξ. Let $(\mathcal{U}_\alpha, \varphi_\alpha)_{\alpha\in A}$ be a $PO(p+1, q+1)$-trivializing atlas for ξ; we assume that the $(\mathcal{U}_\alpha)_{\alpha\in A}$ constitute an atlas with coordinates $(x^\mu)_{\mu=1,2,\ldots,m}$ of N. Let $\{x^\lambda, y^i\}$, with $\lambda = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$, be the system of coordinates of $\bar{M}$ defined on the open set $\varphi_\alpha(\mathcal{U}_\alpha \times \mathcal{U})$, where $(\mathcal{U}, \hat{\varphi})$ is the above local chart (Lemma 2.9.1.3.1), by $(x^\lambda, y^i)(\bar{m}_x) = (x^\lambda(x), e^i \circ \hat{\varphi} \circ \varphi_{\alpha,x}^{-1}(\bar{m}_x))$, $\bar{m}_x \in \bar{M}_x$, $x \in \mathcal{U}_\alpha$.[116]

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[116] e^j denotes the mapping $\sum z^j \varepsilon_j \to z^j$, and the coordinates $y^i(\bar{m}_x)$ are the coordinates of the element y_x in E_n such that $\varphi_{\alpha,x}^{-1}(\bar{m}_x) = \overline{u(y_x)}$, where u is the mapping defined above.

Let $\bar{w}^\alpha_\beta$ be the components in the canonical basis $(e^\alpha_\beta)_{\alpha,\beta=0,1,2,\ldots,n,n+1}$ of the Lie algebra $gl(E_{n+2})$ of the local 1-form $\sigma^*_\alpha(\bar{w})$, where σ_α is the local cross section of the principal associated bundle P_ξ defined by $\sigma_\alpha(x) = \varphi_{\alpha,x}$, $x \in \mathcal{U}_\alpha$, $\sigma^*_\alpha(\bar{w})$ with values in the Lie algebra $po(p+1, q+1)$ isomorphic to the Lie algebra $o(p+1, q+1)$. One can verify the following results:

$$\left.\begin{array}{ll} \bar{w}^0_{n+1} = \bar{w}^{n+1}_0 = 0 & \bar{w}^0_i = -2g_{ij}\bar{w}^j_{n+1} \\ \bar{w}^0_0 + \bar{w}^{n+1}_{n+1} = 0 & \bar{w}^{n+1}_i = -2g_{ij}\bar{w}^j_0 \end{array}\right\}, \qquad i, j = 1, 2, \ldots, n,$$ [117]

$\bar{w}^i_j g_{ik} + g_{ij}\bar{w}^i_k = 0$, $i, j, k = 1, 2, \ldots, n$.

If ϕ^i is the local 1-form on $\varphi_\alpha(\mathcal{U}_\alpha \times \mathcal{U})$ defined by $\phi^i = \Gamma^i_\mu dx^\mu$, where the (Γ^i_μ) are the local components of the Ehresmann connection $H(\bar{M})$ in the system of coordinates (x^μ, y^i), we obtain

$$\phi^i = -\bar{\pi}^*(\bar{w}^i_{n+1}) + \bar{\pi}^*(\bar{w}^i_j)y^j + \pi^*(w^0_j)(\frac{1}{2}y^2 g^{ji} - y^j y^i).$$

Proof. Let $t \to \bar{m}_t$ be a horizontal piecewise differential curve in $\bar{\pi}^{-1}(\mathcal{U}_\alpha)$. Let (φ_t) be the corresponding piecewise differential curve in P_ξ defined by $\varphi_t = \sigma_\alpha(\gamma_t)$, where (γ_t) is the projection on N of the curve $(\bar{m}_t)$. According to 2.8.8.5, the curve $(\bar{z}_t)$ in M_n defined by $\bar{z}_t = \varphi_t^{-1}(\bar{m}_t)$ satisfies the equation

$$\frac{d}{dt}(\bar{z}_t)_{t=t_0} = \frac{d}{dt}(\overline{\exp(t-t_0)A.\bar{z}_{t_0}})_{t=t_0}, \quad \text{with } A = -\sigma^*_\alpha.\bar{w}(\dot{\gamma}_{t_0}), \tag{I}$$

where $A \in o(p+1, q+1)$.

Let (y_t) be the curve in E_n defined by $y_t = \hat{\varphi}(\bar{z}_t)$ such that $\overline{u(y_t)} = \bar{z}_t$. According to (I) let us take t in some reduced neighborhood of t_0. We have

$$\overline{u(y_t)} = \overline{\exp(t-t_0)A}.\overline{u(y_{t_0})},$$

and therefore

$$\overline{u(y_t)} = \overline{u(f_t(y_{t_0}))} \quad \text{with } f_t = h_1(\overline{\exp(t-t_0)A}),$$

where h_1 *is the isomorphism from* $PO(p+1, q+1)$ *onto* $C_n(p,q) = \text{Conf}\,(E_n(p,q))$ *defined in 2.5.1.2* (cf. below, exercises). Therefore, $y_t = f_t(y_t)$ with $f_t = h_1(\exp(t - t_0)A)$, whence $\dot{y}_{t_0}$ is a conformal infinitesimal transformation of $E_n(p,q)$. We give now the following statement before concluding.

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[117] The fundamental bilinear symmetric form B on $F = E_{n+2}(p+1, q+1) = E_n(p,q) \oplus H = E_n(p,q) \oplus E_2(1,1)$ is defined by $B_{n+2}(x,y) = B(x,y)$ for x, y in $E_n(p,q)$, by O for $x \in E_n(p,q)$, $y \in E_2(1,1)$, O for $x \in E_2(1,1)$, $y \in E_n(p,q)$, and $B_2(x,y)$ for any x, y in H. (B_2 is the standard usual scalar product on the hyperbolic plane H.) We put for any x in $E_n(p,q)$, $x^2 = g_{ij}x^i x^j$, as usual.

Table 2.1.

Elements of $o(p+1, q+1)$ written in the basis $\{f_{i\,j}, f_{0\,n+1}, f_{i\,0}, f_{i\,n+1}\}$	Associated infinitesimal conformal transformations
$f_{ij} = e_{ij} - e_{ji} = g_{ik}e^k_j - g_{jk}e^k_i$ $1 \le i \le j \le n$ basis of the Lie algebra $o(p, q)$	$E_{ij} = x_i \frac{\partial}{\partial x^j} - x_j \frac{\partial}{\partial x^i}$ (corresponding to elements of $O(p, q)$)
$2f_{n+1\,0} = 2(e_{n+1\,0} - e_{0\,n+1}) = e^0_0 - e^{n+1}_{n+1}$	$E_0 = x_i \frac{\partial}{\partial x^i}$ (corresponding to dilations)
$2f_{i\,0} = 2(e_{i\,0} - e_{0\,i}) = -e^{n+1}_i + 2g_{ik}e^k_0$	$E_i = \frac{\partial}{\partial x^i}$ (corresponding to translations)
$2f_{n+1\,i} = 2(e_{n+1\,i} - e_{i\,n+1}) = e^0_i - 2g_{ik}e^k_{n+1}$	$F_i = x^2 \frac{\partial}{\partial x^i} - 2x_i x^k \frac{\partial}{\partial x^k}$ (corresponding to transversions or special conformal translations)

2.9.1.3.3 Table of Infinitesimal Conformal Transformations of $E_n(p, q)$

(Infinitesimal version of the table given in 2.4.2.4) The isomorphism h_1 given in 2.5.1.2 from $E_n(p, q)$ onto $C_n(p, q)$ leads to an isomorphism H_1 from the Lie algebra $po(p+1, q+1)$ onto the Lie algebra of conformal infinitesimal transformations of $E_n(p, q)$ classically defined as the Lie algebra of vectors fields X on $E_n(p, q)$ such that $L_X q = \mu_X q$, $\mu_X q$ being a scalar, where L_X denotes the Lie derivative by the vector field X and q the fundamental quadratic form on $E_n(p, q)$.[118]

Let $\{(f_{i\,j}, f_{0\,n+1}, f_{i\,0}, f_{i\,n+1})_{1 \le i < j \le n}\}$ be the basis of the Lie algebra of $o(p+1, q+1)$ with $f_{\alpha\beta} = e_{\alpha\beta} - e_{\beta\alpha}$; $\alpha, \beta = 0, 1, 2, \ldots, n, n+1$, where $e_{\alpha\beta} = g_{\alpha\gamma}e^\gamma_\beta$ and where e^α_β is the canonical basis of the Lie algebra $gl(E_{n+2})$.

We can easily obtain Table 2.1 (cf. below exercises).

The $(n+1)(n+2)/2$ elements (E_{ij}, E_0, E_i, F_i) constitute a basis[119] of the Lie algebra of infinitesimal conformal transformations of $E_n(p, q)$. One can easily verify that

$$[E_{ij}, E_{kl}] = g_{jk}E_{il} + g_{il}E_{jk} - g_{jl}E_{ik} - g_{ik}E_{jl} \text{ (table of the Lie algebra } o(p, q)),$$
$$[E_i, F_j] = 2E_{ij} - 2g_{ij}E_0,$$

118 Let us recall the following fact (cf., for example, Yvette Kosmann, *C.R. Acad. Sc. Paris*, t. 280, 27 Janvier 1975, serie A, pp. 229–232). Let $\mathcal{G}$ be the Lie algebra of infinitesimal conformal transformations of V_n, an n-dimensional riemannian or pseudo-riemannian manifold V_n, whose tensor metric is denoted by g. $\mathcal{G}$ is the Lie algebra of infinitesimal conformal transformations of V_n, that is, of vector fields X on V_n such that $\mathcal{L}(X)g = -2(\delta X/n)g$, where $\delta(X)$ is the divergence of the vector field X and $\mathcal{L}(X)$ the Lie derivative by X. $\dim \mathcal{G} \le (n+1)(n+2)/2$ and $\dim \mathcal{G} = (n+1)(n+2)/2$ if V_n is a pseudo-Euclidean or Euclidean vector space or a sphere. If V_n is the flat standard Minkowski space dim, $\mathcal{G} = 15$. For a standard pseudo-Euclidean space of type (p, q), $E_n(p, q)$ $\mathcal{G}$ is isomorphic to so $(p+1, q+1) \simeq o(p+1, q+1)$.

119 Such a classical result is given, for example, in A. Crumeyrolle, Fibrations spinorielles et twisteurs généralisés, *Periodica Math. Hungarica*, vol. 6.2, 1975, pp. 143–171.

$$\begin{aligned}
[E_i, E_j] &= [F_i, F_j] = 0,\\
[E_i, E_0] &= E_i,\\
[F_i, E_0] &= -F_i,\\
[E_{ij}, E_0] &= 0,\\
[E_{ij}, E_k] &= -g_{ik}E_j + g_{jk}E_i,\\
[E_{ij}, F_k] &= -g_{ik}F_j + g_{jk}F_i.
\end{aligned}$$

Thus, the Lie algebra $po(p+1, q+1)$ is isomorphic to the Lie algebra $\mathbf{R}^n \oplus co(p,q) \oplus (\mathbf{R}^n)^*$, where $co(p,q)$ denotes the Lie algebra associated with the classical group $CO(p,q)$ of similarities of $E_n(p,q)$.[120]

One can easily verify this result in the following way: the Lie subalgebra generated by (E_{ij}, E_0) is identified with $o(p,q) \oplus \mathbf{R}$; the Lie subalgebra generated by (E_i) is identified with $\mathbf{R}^n$, by identifying E_i and e_i; the Lie subalgebra generated by (F^i) is identified with $(\mathbf{R}^n)^*$ by identifying

$$F^i = \left(\frac{1}{2}x^2 g^{ij} - x^i x^j\right)\frac{\partial}{\partial x^j} \quad \text{with } e^i.$$

2.9.1.3.4 Remark Since the action of $PO(p+1, q+1)$ on the Möbius space M_n is transitive, all the groups of isotropy are isomorphic one to the other.

We will use this result later.

Now we can achieve the proof of 2.9.1.3.2.

According to the table, given 2.9.1.3.3, we can write

$$\frac{dy^i}{dt} = \bar{w}^i_{n+1}(\dot{\gamma}_t) - \bar{w}^i_j(\dot{\gamma}_t)y^j - \bar{w}^0_0(\dot{\gamma}_t)y^i + \bar{w}^{n+1}_j(\dot{\gamma}_t)(\frac{1}{2}y^2 g^{ji} - y^j y^i),$$

whence we can deduce the result given above, since the coordinates $y^i(\bar{m}_t)$ of the horizontal curve $(\bar{m}_t)$ satisfy the equations given in 2.8.7.3, $dy^i/dt = -\Gamma^i_\lambda(p_t)(dx^\lambda/dt)$ and since $\phi^i = \Gamma^i_\mu dx^\mu$.

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[120] We recall that the "generalized" Lorentz group $O(p,q)$ has $\frac{n(n-1)}{2}$ parameters, the group of translations $\mathcal{T}_n$ has n parameters, the Poincaré group $P(p,q)$ semidirect product of $O(p,q)$ and $\mathcal{C}_n$ has $\frac{n(n+1)}{2}$ parameters. The group $CO(p,q)$ of similarities of $E_n(p,q)$ if $p \neq q$ is the direct product of $O(p,q)$ and $\mathbf{R}^+$, the group of positive dilatations (or dilations). If $p = q$ and then E_n is of even dimension, $CO^+(p,q)$ is normal and of index 2 in $CO(p,q)$ and $CO^+(p,q)$ the group of positive (or direct) similarities is the direct product of $O(p,q)$ and $\mathbf{R}^+$. If $p \neq q$, $CO(p,q)$ has $\frac{n(n-1)}{2} + 1$ parameters and if $p = q$, $CO^+(p,q)$ has $\frac{n(n-1)}{2} + 1$ parameters. The group of conformal affine transformations of E_n has $\frac{n(n+1)}{2} + 1$ parameters.

Conf $(E_n(p,q)) = C_n(p,q)$ has $\frac{(n+1)(n+2)}{2}$ parameters. Its Lie algebra *Lie*$(C_n(p,q))$ is isomorphic with $po(p+1, q+1)$.

The notation $CO(p,q)$ is used by S. Kobayashi, Transformations groups in differential geometry, op.cit., p. 10, for example, to denote the group of similarities. The corresponding notation used by J. Dieudonné, La géomètrie des groupes classiques, op. cit., for the same group is $GO(p,q)$. Cf. also 2.4.2.6.2 above.

2.9.1.3.5 Remark If we decompose an element A in $po(p+1,q+1)$, identified with $o(p+1,q+1)$ and defined in the canonical basis $(e^\alpha_\beta)_{\alpha,\beta=0,1,2,\ldots,n,n+1}$ of $gl(E_{n+2})$ as $A = A^\alpha_\beta e^\beta_\alpha$, we can write

$$A = B^i e_i + B^i_j e^j_i + B^0_i e_i \quad \text{with } B^i = -A^i_{n+1},\ B^i_j = A^i_j + A^0_0 \delta^i_j,\ B^0_i = -A^{n+1}_i.$$

The local 1–form $\sigma^*_\alpha(\bar{w})$ can be written as $\sigma^*_\alpha(\bar{w}) = w^i e_i + w^i_j e^j_i + w^0_i e^i$ with $w^i = -\bar{w}^i_{n+1}$, $w^i_j = \bar{w}^i_j + \bar{w}^0_0 \delta^i_j$, $w^0_i = -\bar{w}^{n+1}_i$. Therefore the preceding result given in 2.9.1.3.2 can now be written

$$\phi^i = \bar{\pi}^*(w^i) + \bar{\pi}(w^i_j) y^j + \pi^*(w^0_j)\left(\frac{1}{2} y^2 g^{ji} - y^j y^i\right).$$

We give now three propositions (see exercises below).[121]

2.9.1.4 Proposition *Let $H(\bar{M})$ be a conformal Ehresmann connection on a fiber bundle $\xi = (\bar{M}, \bar{\pi}, N, M_n, PO(p+1,q+1))$. With the same notation as above, the local 1-forms of the Ehresmann connection $\phi^i = \Gamma^i_\lambda dx^\lambda$ satisfies the relation*

$$\phi^i = \bar{\pi}^*(w^i) + \pi^*(w^i_j) y^j + \bar{\pi}^*(w^0_j)\left(\frac{1}{2} y^2 g^{ji} - y^j y^i\right)$$

with $\{w^i, w^i_j, w^0_j\}$ being $(n+1)(n+2)/2$ local 1-forms on N such that $w^i_j g_{ik} + g_{ji} w^i_k = (2/n) \sum_k w^k_k g_{jk}$.

2.9.1.5 Proposition *Let $H(M_\eta)$ be a projective Ehresmann connection on a fiber bundle $\eta = (M_\eta, \pi_\eta, N, P(E_{n+2}), PO(p+1,q+1))$. With the same notation as above, the local 1-forms $\tilde{\phi}^0 = \tilde{\Gamma}^0_\lambda dx^\lambda$, $\tilde{\phi}^i = \tilde{\Gamma}^i_\lambda dx^\lambda$, where $(\tilde{\Gamma}^0_\lambda, \tilde{\Gamma}^i_\lambda)$ are the local components of the Ehresmann connection $H(M_\eta)$, satisfy the relations*

$$\tilde{\phi}^i = \pi^*_\eta(\sigma^i) + \pi^*_\eta(\sigma^i_j) z^j + \pi^*_\eta(\sigma^0_j)\left(\frac{1}{2} g^{ji} z^0 + z^j z^i\right),$$

$$\tilde{\phi}^0 = \frac{2}{n} \sum_k \pi^*_\eta(\sigma^k_k) z^0 - 2 z_j \pi^*_\eta(\sigma^j) - \pi^*_\eta(\sigma^0_k) z^k z^0,$$

where the $(\sigma^i, \sigma^i_j, \sigma^0_j)$ are $(n+1)(n+2)/2$ local 1-forms on N such that $\sigma^i_j g_{ik} + g_{ji} \sigma^i_k = (2/n) \sum_k \sigma^k_k g_{jk}$.

2.9.1.6 Proposition *Let P be a principal bundle with base space N and structure group $PO(p+1,q+1)$ and let $H(P)$ be a principal connection on P. Let η,*

[121] These results are due to J. L. Milhorat, Sur les connections conformes, Thesis, Université Paul Sabatier, 1986.

respectively ξ, be the corresponding fiber bundle with typical fiber $P(E_{n+2})$, respectively M_n, and with the same structure group $PO(p+1, q+1)$ as defined above. Let $H(M_\eta)$, respectively $H(\bar{M})$, be the connection on η, respectively ξ, associated with the principal connection $H(P)$. Let j denote the identical mapping from $\bar{M}$ into M_η (locally the inclusion $M_n \subset P(E_{n+2})$). Then j is an embedding and satisfies the relation $j^*(H(\bar{M})) = j^*(T(\bar{M})) \cap H(M_\eta)$.

2.9.2 Cartan Conformal Connections

2.9.2.1 Classic Cartan Conformal Connections[122]

2.9.2.1.1 Jets and r-Frames

Let M be a manifold of dimension n.

2.9.2.1.2 Definition Let M be a manifold of dimension $n \geq 3$. Let $\mathcal{V}$ be the set of open neighborhoods of 0 in $\mathbf{R}^n$. Let f and g be respectively two mappings, $f : U \to M$, $g : V \to M$, where $U, V \in \mathcal{V}$. f and g are said to define the same r-jet at 0 if $f(0) = g(0)$ and if there exists a local chart (Ω, h) of M at $a = f(0) = g(0)$ such that the mappings $h \circ f : U \to \mathbf{R}^n$ and $h \circ g : V \to \mathbf{R}^n$ have the same partial derivatives up to order r at 0.

The same is true for any other chart (Ω', h') at a. Thus the relation "f and g define the same r-jet at 0" is an equivalence relation on the set of mappings such that $f : U \to M$, with $U \in \mathcal{V}$. Any equivalence class is denoted by $j_0^r(f)$ and called an r-jet at 0. If $f : U \to M$ is a diffeomorphism from an open neighborhood of 0 onto an open subset of M, the r-jet $j_0^r(f)$ is called an r-frame at $a = f(0)$ Then $(f(U), f^{-1})$ is a local chart of M at a.

2.9.2.1.3 Proposition *The set of r-frames of M, denoted by $P^r(M)$, is a principal bundle over M with projection $p : P^r(M) \to M$, the natural projection defined by $p(j_0^r(f)) = f(0)$ that sends any r-frame onto its origin.*

The structure group $G^r(n)$ is the set of r-frames $j_0^r(\varphi)$ where $\varphi : U \to \mathbf{R}^n, U \in \mathcal{V}$, is a diffeomorphism such that $\varphi(0) = 0$, provided with the following composition of jets, namely

$$(j_0^r(\varphi'), j_0^r(\varphi)) \to j_0^r(\varphi' \circ \varphi).$$

$G^r(n)$ acts on $P^r(M)$ on the right by

$$j_0^r(f).j_0^r(\varphi) = j_0^r(f \circ \varphi) \quad \text{for } j_0^r(f) \in P^r(M) \text{ and } j_0^r(\varphi) \in G^r(n).$$

[122] Most of the following results can be found on pp. 127–149 in the following book of reference: S. Kobayashi, *Transformations Groups in Differential Geometry*, Springer-Verlag, 1972; and in the following thesis: A. Toure, Divers aspects des connections conformes, Thesis, Université de Paris VI, 1981. Cf. also R. Hermann, *Vector Bundles in Mathematical Physics*, vol. 1, W.A. Benjamin, Inc., New York, 1970, chapter II.

The proof[123] is straightforward. The Lie algebra of $G^r(n)$ will be denoted by $g^r(n)$.

2.9.2.1.4 Examples

1. $P^1(M)$ is the bundle of linear frames over M with structure group $G^1(n) = GL(n, \mathbf{R})$.
2. $P^1(\mathbf{R}^n)$ can be identified with the group $\mathcal{A}(n, \mathbf{R})$ of affine bijections of $\mathbf{R}^n$ whose Lie algebra is $\mathbf{R}^n \oplus gl(n, \mathbf{R})$. $P^1(\mathbf{R}^n)$ is a principal bundle with base $\mathbf{R}^n$ and structure group $GL(n, \mathbf{R})$. The neutral element of $\mathcal{A}(n, \mathbf{R})$ will be denoted by e.

In the same way as above, one can consider for U, a given open set of $\mathbf{R}^n$, the mapping $H_U : P^r(U) \to \mathbf{R}^n \times G^r(n)$ defined by $H_U(j_0^r(f)) = (f(0), j_0^r(f - f(0)))$.

It is a bijective mapping that provides $P^r(U)$ with the structure of a product of manifolds. The result is true for $U = \mathbf{R}^n$. $P^r(U)$ is an open set of $P^r(\mathbf{R}^n)$. If $f : U \to M$ is a diffeomorphism, the mapping $\tilde{f}_r : P^r(U) \to P^r(f(U))$, $j_0^r(\varphi) \to j_0^r(f \circ \varphi)$ satisfies $\tilde{f}_r(e) = j_0^r(f)$ and is a bijective mapping that allows the transfer of the structure of product of manifolds onto $P^r(f(U))$. Thus, by varying the chart $(f(U), f^{-1})$ we can obtain the structure of a fiber bundle of $P^r(M)$. Then, $\tilde{f}_r$ appears as an isomorphism from the bundle $P^r(U) = p^{-1}(U)$, which is an open set of $P^r(\mathbf{R}^n)$- onto $P^r(f(U)) = p^{-1}(f(U))$, which is an open set of $P^r(M)$.

2.9.2.1.5 Study of $P^2(M)$

2.9.2.1.5.1 Local Coordinates Let $(e_i)_{i=1,2,\ldots,n}$ be the natural basis for $\mathbf{R}^n$ and $(x^1, \ldots, x^n)$ the natural system of coordinates in $\mathbf{R}^n$. Any element $u = j_0^2(\varphi)$ of $P^2(\mathbf{R}^n)$ is defined by the polynomial representation

$$\varphi(x) = \sum_i \left(u^i + \sum_j u^i_j x^j + \frac{1}{2} \sum_j \sum_k u^i_{j\,k} x^j x^k \right) e_i,$$

where

$$x = \sum_i x^i e_i$$

and $u^i_{j\,k} = u^i_{k\,j}$

2.9.2.1.5.2 Definitions $(u^i, u^i_j, u^i_{j\,k})$ are called the natural canonical coordinates of $u = j_0^2(\varphi)$, and we write simply $j_0^2(\varphi) = (u^i, u^i_j, u^i_{j\,k})$. By restriction, $G^2(n)$ is constituted of elements $(0, a^i_j, a^i_{j\,k})$ simply written $a = (a^i_j, a^i_{j\,k})$. The right action

[123] It is sufficient to remark that the right action defined above is simply transitive on any of the fibers $p^{-1}(a)$, where $a \in M$, since if $f : U \to M$ and $g : V \to M$ are two diffeomorphisms such that $f(0) = g(0) = a$, then $\varphi = f^{-1} \circ g$ is a diffeomorphism such that $j_0^r(g) = j_0^r(f) j_0^r(\varphi)$.

of $G^2(n)$ onto $P^2(\mathbf{R}^n)$ is defined by the mapping $P^2(\mathbf{R}^n) \times G^2(n) \to P^2(\mathbf{R}^n)$ that sends $(j_0^2(\varphi), a) = (u, a)$ into $j_0^2(\varphi \circ \alpha)$ if we set $a = j_0^2(\alpha)$, i.e.,

$$(u^i, u^i_j, u^i_{jk}) * (a^i_j, a^i_{jk}) = \left(u^i, \sum_r u^i_r a^r_j, \sum_r u^i_r a^r_{jk} + \sum_r u^i_{rs} a^r_j a^s_k\right).$$ [124]

2.9.2.1.5.3 Canonical Form on $P^2(M)$: Definition We want to define a 1-form with values in the Lie algebra of $\mathcal{A}(n, \mathbf{R})$, namely $\mathbf{R}^n \oplus gl(n, \mathbf{R})$, which will be described. Let X be a tangent vector to $P^2(M)$ at a point $u = j_0^2(f)$. Let X' be its image via the canonical projection $P^2(M) \to P^1(M)$ that sends $j_0^2(g)$ into $j_0^1(g)$. X' is a tangent vector to $P^1(M)$ at $j_0^1(f)$. From the results given above, we can deduce that f induces an isomorphism $\tilde{f}_1$ from an open set $P^1(U)$ of $P^1(\mathbf{R}^n)$ onto the open set $P^1(f(U))$ of $P^1(M)$; here, U denotes the set of definition for f. We know that $\tilde{f}_1(e) = j_0^1(f)$. There exists a unique tangent vector Y to $P^1(\mathbf{R}^n)$ at e such that $\tilde{f}_1'(e).Y = X'$. Since the tangent space to $P^1(\mathbf{R}^n) = \mathcal{A}(n, \mathbf{R})$ at e can be identified with the Lie algebra of $\mathcal{A}(n, \mathbf{R})$, Y takes its values in $\mathbf{R}^n \oplus gl(n, \mathbf{R})$.

By definition we put $\theta(X) = Y$, and thus define a 1-form θ, and the value of θ at u depends only on $j_0^2(f) = u$.

On the other hand, if $h : P^2(M) \to P^2(\bar{M})$ is a morphism of fiber bundles and if θ and $\bar{\theta}$ denote respectively the canonical forms of $P^2(M)$, respectively $P^2(\bar{M})$, we have

$$h^*(\bar{\theta}) = \theta.$$

In particular, if $f : U \to M$ is a diffeomorphism that defines local coordinates and $\tilde{f}_2 : P^2(U) \to P^2(M)$ is the corresponding morphism of fiber bundles associated with f, we have $\tilde{f}_2^*(\theta) = \bar{\theta}$, where $\bar{\theta}$ denotes the canonical form of $P^2(\mathbf{R}^n)$.

Such a form $\bar{\theta} = \tilde{f}_2^*(\theta)$ will be called an expression of θ in the system of local coordinates defined by f on $P^2(M)$. In order to determine such an expression, it is sufficient to calculate the form θ, denoted here by $\bar{\theta}$, in the case $M = \mathbf{R}^n$.

With the same notation as above, let $u = j_0^2(f)$ be an element of $P^2(\mathbf{R}^n)$. We put $u = (u^i, u^i_j, u^i_{jk})$ with $u^i = f^i(0)$, $u^i_j = (\partial f^i/\partial x^j)(0)$, and $u^i_{jk} = (\partial^2 f^i/\partial x^j \partial x^k)(0)$. If $y = j_0^1(\varphi) = (y^j, y^i_j)$ is an element of $P^1(\mathbf{R}^n)$, we have

$$\tilde{f}_1(y) = j_0^1(f \circ \varphi) = \left(f^1(y^1, \ldots, y^n), \sum_k \frac{\partial f^i}{\partial y^k}(y^1, \ldots, y^n) y^k_j\right).$$

[124] We note that the multiplication in $G^2(n)$ is given by

$$(b^i_j, b^i_{jk})(a^i_j, a^i_{jk}) = \left(\sum_k b^i_k a^k_j, \sum_r b^i_r a^r_{jk} + \sum_r b^i_{rs} a^r_j a^s_k\right).$$

Thus by differentiation at $y = e = (0, \delta^i_j)$,

$$f_1'(e).dy = \left(\sum_k \frac{\partial f^i}{\partial y^k}(0)dy^k, \sum_k \frac{\partial^2 f^i}{\partial y^k \partial y^j}(0)dy^k + \sum_k \frac{\partial f^i}{\partial y^k}(0)dy^k_j\right)$$
$$= \left(\sum_k u^i_k dy^k, \sum_k u^i_{k\,j} dy^k + \sum_k u^i_k dy^k_j\right).$$

Let us denote by $E_i = \partial/\partial y^i$, $E^j_i = \partial/\partial y^i_j$ the canonical basis of $\mathbf{R}^n \oplus gl(n, \mathbf{R})$—the Lie algebra of $G^2(n)$.

Any tangent vector X to $P^2(\mathbf{R}^n)$ at $u = j^2_0(f)$ can be written in the system of coordinates $(u^i, u^i_j, u^i_{j\,k})$:

$$X = \sum_i \frac{\partial}{\partial u^i} X^i + \sum_{i,j} \frac{\partial}{\partial u^i_j} X^i_j + \sum_{j,k,i} \frac{\partial}{\partial u^i_{j\,k}} X^i_{j\,k}.$$

The image of X by the canonical projection: $P^2(M) \to P^1(M)$ is then given by

$$X' = \sum_i \frac{\partial}{\partial u^i} X^i + \sum_{i,j} \frac{\partial}{\partial u^i_j} X^i_j.$$

Solving these equations in Y^k, Y^k_j, we obtain, from the definition,

$$\bar\theta(X) = Y^i = \sum_k v^i_k X^k, \bar\theta^i_j(X) = Y^i_j = \sum_k v^i_k X^k_j - \sum_{k,h,l} v^i_k u^k_{h\,j} v^h_l X^l,$$

where (v^i_k) denotes the inverse matrix of (u^i_k). We write simply

$$\bar\theta^i = \sum v^i_k du^k, \bar\theta^i_j = \sum v^i_k du^k_j - \sum_{k,h,l} v^i_k u^k_{h\,j} v^h_l du^l,$$

whence we deduce

$$d\bar\theta^i = -\sum_k \theta^i_k \wedge \theta^k,$$

since $d(u^i_k v^k_j) = 0 \Rightarrow u^i_k dv^h_j = -du^i_k v^k_j = -\sum v^h_i du^i_k v^k_j$,

$$d\bar\theta^i = \sum_k dv^i_k \wedge du^k = -\sum_{k,h,l} v^i_h du^k_l v^l_k \wedge du^k = -\sum_{h,l} v^i_h du^h_l \wedge \bar\theta^k.$$

On the other hand,

$$\sum_{k,h,l} v^i_k u^k_{hj} v^l_h du^l \wedge \bar\theta^j = \sum_{k,h,l} v^i_k u^k_{hj} \bar\theta^h \wedge \bar\theta^j = 0,$$

taking account that $u^k_{j\,h} = u^k_{h\,j}$. Then,

$$d\bar\theta^i = -\sum_{h,l} v^i_h du^h_l \wedge \bar\theta^l = -\sum \bar\theta^i_l \wedge \bar\theta^l.$$

We have obtained the following result:

2.9.2.1.5.4 Proposition[125] *Let* $\theta = (\theta^i, \theta^i_j)$ *be the canonical form on* $P^2(M)$. *Then* $d\theta^i = -\sum \theta^i_k \wedge \theta^k$.

2.9.2.1.5.5 Fundamental Vector Fields We know that $G^r(n)$ acts on the right on $P^r(M)$ according to the following law: $P^r(M) \times G^r(n) \rightarrow P^r(M)$ defined by $(u, g) \rightarrow R_g(u)$. With any element L in the Lie algebra $g^r(n)$ of $G^r(n)$ we can associate a fundamental vector field L^* on $P^r(M)$ defined in 2.8.1.1.

For any $a \in G^r(n)$, the vector field $R^*_a L^*$ is the field X defined on $P^r(M)$ by $X(R_a(u)) = dR_a(u) \cdot L^*(u)$ (often denoted $R'_a(u) \cdot L^*(u)$). Since $(R_{\exp tL})_{t \in \mathbf{R}}$ is the 1-parameter group generated by L^*, we can deduce that the 1-parameter group generated by $X = R^*_a L^*$ is $(R_{a^{-1}\exp(tL)_a})_{t \in \mathbf{R}}$. Since $a^{-1}\exp(tL)a = \exp(tM)$ with $M = ad(a^{-1})L$, we find that $R^*_a L^* = (ad(a^{-1})L)^*$.

The fundamental vector fields are vertical, i.e., tangent to the fiber. Moreover, if X is a fundamental vector field on $P^2(M)$, namely $X = L^*$, with $L \in g^2(n)$, the fundamental form θ satisfies $\theta(X(u)) = L'$ for any $u \in P^2(M)$, where L' stands for the canonical projection of L onto $g^1(n) = \mathfrak{gl}(n, \mathbf{R})$ that can be identified with $\mathrm{Hom}(\mathbf{R}^n)$, the space of morphisms from $\mathbf{R}^n$ to $\mathbf{R}^n$.

2.9.2.2 *G*-Structures and Conformal Structure

Let M be a n-dimensional differentiable paracompact manifold.

2.9.2.2.1 Introductory Notes

2.9.2.2.1.1 Definition Let H be a closed subgroup of $G^r(n)$. A reduction to H of the structure group $G^r(n)$ of $P^r(M)$ is a principal subbundle of $P^r(M)$ with structure group H. Simply, we say that such a bundle is an H-reduction of $P^r(M)$. It is given by the datum of an open covering $(\mathcal{U}_\alpha)_{\alpha \in A}$ of M and a family $(S_\alpha)_{\alpha \in A}$ of local cross sections such that:

(i) $S_\alpha : \mathcal{U}_\alpha \rightarrow P^r(M)$ defined by $x \rightarrow j^r_0(S^x_\alpha)$ where the function S^x_α satisfies $S^x_\alpha(0) = x$ such that for any (α, β, x) with $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta$ there exists an element $a^x_{\alpha\beta}$ in H such that

(ii) $S_\beta(x) = S_\alpha(x) a^x_{\alpha\beta}$. (The group H acts on the right on $P^r(M)$.)

The reduced bundle is then defined by the morphisms $h_\alpha : \mathcal{U}_\alpha \times H \rightarrow P^r(M)$ that send (x, h) into $S^x_\alpha h$, with transition functions $a^x_{\alpha\beta}$. Two families $(\mathcal{U}_\alpha, S_\alpha)_{\alpha \in A}$, $(\mathcal{U}_i, S_i)_{i \in A}$ of local cross sections define the same H-reduction if their union satisfies (ii) above.

[125] This result is given in the following book: S. Kobayashi, *Transformations Groups in Differential Geometry*, Springer, 1972, p. 141, Proposition 5.2.

Since H acts on the right on $P^r(M)$, the quotient space $P^r(M)/H$ is a bundle associated with $P^r(M)$, with typical fiber $G^r(n)/H$ (right-quotient), and condition (ii) is equivalent to $S_\alpha(x)$ and $S_\beta(x)$ belong the same equivalence class modulo H, namely $\bar{S}_\alpha(x) = \bar{S}_\beta(x)$. The datum of local cross section S_α satisfying (ii) is equivalent to the datum of a cross section $\sigma : x \to \bar{\sigma}_\alpha(x)$ of the bundle $P^r(M)/H$. Since M is a paracompact manifold, a sufficient condition for the existence of such a cross section is that $G^r(n)/H$ be homeomorphic to a standard Euclidean space $\mathbf{R}^p$ that is a solid space.[126]

2.9.2.2.1.2 Proposition Extension of a Reduction *Let G, H be two closed subgroups of $G^r(n)$ such that $G \subset H$ and let $G(M)$ be a G-reduction of $P^r(M)$. There exists an H-reduction canonically associated with $G(M)$ such that $G(M)$ is an H-reduction.*

Let assume that $G(M)$ is defined by the cross section $\sigma : M \to P^r(M)/G$ and let π denote the canonical projection from $P^r(M)/G$ onto $P^r(M)/H$ that sends any class modulo G onto the class modulo H that contains it. Then $\pi \circ \sigma : M \to P^r(M)/H$ is a cross section that defines an H-reduction determined by the datum of $G(M)$, namely $H(M)$.

We need to notice that the datum of $H(M)$ does not determine $G(M)$. More precisely, with the above notation, we have the following result.

2.9.2.2.1.3 Proposition *Two G-reductions of $P^r(M)$ respectively defined by the sections σ and σ' of $P^r(M)/G$ determine the same H-reduction $H(M)$ if and only if for any $x \in M$, $\sigma(x)$ and $\sigma'(x)$ are in the same class modulo H.*

The proof is left as an exercise.

---

[126] We recall the following definition and properties (N. Steenrod, *The Topology of Fiber Bundles*, Princeton University Press, 1951, pp. 54–56.) A space Y will be called solid if for any normal space X, a closed subset A of X, and map $f : A \to Y$, there exists a map $f' : X \to Y$ such that $f'|A = f$.

Examples: $\mathbf{R}^n$, a Euclidean n-space is a solid space. Let X be a normal space with the property that every covering of X by open sets is reducible to a countable covering (e.g., X is compact, or has a countable basis). Let A be closed in X. Let G be a Lie group and H a closed subgroup such that G/H is solid. Then any (G, H)-bundle over (X, A) is (G, H)-equivalent to an (H, H) bundle.

Corollary: With the same assumptions, any bundle over X with group G is equivalent in G to a bundle with group H.

A normal space is a topological space that is a separated space that satisfies the following property: any pair of closed subsets F, F', with $F \cap F' = \emptyset$ possesses a pair of neighborhoods $\mathcal{V}$, for F, $\mathcal{V}'$, for F', such that $\mathcal{V} \cap \mathcal{V}' = \emptyset$. A separated space is normal if each pair of disjoint closed sets have disjoint neighborhoods.

If we suppose that σ and σ' are respectively defined by families of local cross sections $(\mathcal{U}_\alpha, S_\alpha)$, $(\mathcal{U}'_\beta, S'_\beta)$ of $P^r(M)$, the result means that for any α, β, x such that $x \in \mathcal{U}_\alpha \cap \mathcal{U}'_\beta$, there exists $b^x_{\alpha\beta}$ in H such that $S'_\beta(x) = S_\alpha(x) b^x_{\alpha\beta}$.

2.9.2.2.1.4 Definition Let $P^1(M)$ be the principal bundle of 1-frames of M, with structure group $GL(n, \mathbf{R})$, and let G be a subgroup of $GL(n, \mathbf{R})$. A G-structure on M is a subbundle of $P^1(M)$ with structure group G, namely a restriction to G of the structure group of $P^1(M)$.

2.9.2.2.1.5 Proposition *Such a G-structure exists if and only if the associated bundle with $P^1(M)$ with typical fiber $GL(n, \mathbf{R})/G$ admits a section. Since we assume that M is paracompact, such a section exists if the quotient $GL(n, \mathbf{R})/G$ is diffeomorphic to a Euclidean space $\mathbf{R}^p$.*

2.9.2.2.1.6 Examples–Definitions A riemannian structure on M is an $O(n)$-structure. A pseudo-riemannian structure on M is an $O(p, q)$-structure. A conformal structure on M is a $CO(n)$-structure, where

$$CO(n) = \{A \in GL(n, \mathbf{R}) : {}^t A g A = \rho g, \rho \in \mathbf{R}^+\},$$

with Lie algebra

$$co(n) = \{A \in gl(n, \mathbf{R}) : {}^t A g + g A = \rho g, \rho \in \mathbf{R}\}.$$

A generalized conformal structure on M is a $CO(p, q)$-structure, where $CO(p, q)$ stands for the group of similarities of $E_n(p, q)$, $CO(p, q) = O(p, q) \times \mathbf{R}^{+*}$, and the Lie algebra $po(p + 1, q + 1)$ is isomorphic $\mathbf{R}^n \oplus co(p, q) \oplus (\mathbf{R}^n)^*$ where $co(p, q)$ denotes the Lie algebra of $CO(p, q)$. (Cf. above, 2.9.1.3.2 and footnote 128.)

According to 2.9.2.1.2.2, the datum of a riemannian, respectively a pseudo-riemannian, structure implies that of a conformal, respectively generalized conformal, structure. Conversely, since $\frac{CO(n)}{O(n)}$, respectively $\frac{CO(p,q)}{O(p,q)}$, is diffeomorphic to $\mathbf{R}^{+*}$ and then to $\mathbf{R}$, any conformal structure, respectively generalized conformal structure, is reducible to a riemannian structure, respectively a pseudo-riemannian structure. But such a reduction is not unique (cf. 2.9.2.2.1.5).

2.9.2.2.1.7 Equivalent Definitions We consider only the case of the usual Möbius group (2.2), $\tilde{M}(n)$. Since the Möbius classical group acts transitively on the Möbius space M_n, all the groups of isotropy are isomorphic one to the other. Let $\tilde{M}_0(n)$ denote the group of isotropy of the origin for the standard Möbius group $\tilde{M}(n)$.

One can easily identify[127] $\tilde{M}_0(n)$ with the closed subgroup of $G^2(n)$ consisting of jets $j^2_0(\varphi)$ such that $\varphi(0) = 0$ and $\varphi'(x) \in CO(n)$, for x in some neighborhood of 0. Let us consider $\tilde{M}_1(n)$, the subgroup of $\tilde{M}_0(n)$ consisting of jets such that $\varphi'(0) = \mathrm{Id}$.

[127] The proof is left as an exercise. (cf. below 2.13, Exercise XV.1.)

One can see that $\tilde{M}_1(n)$ is a normal subgroup of $\tilde{M}_0(n)$ (isomorphic to the group of the translations of the standard space E_n) and that $\frac{\tilde{M}_0(n)}{\tilde{M}_1(n)}$ is isomorphic to $CO(n)$. Thus if P is a subbundle of $P^2(M)$ with structure group $\tilde{M}_0(n)$, the homogeneous space $P/\tilde{M}_1(n)$ is a principal bundle with typical fiber $CO(n)$ and thus defines a conformal structure on M.

Conversely, let $Q(M)$ be a subbundle of $P^1(M)$ with typical fiber $CO(n)$ that defines a conformal structure M. According to the general theory of prolongations of G-structures,[128] one can associate with it a subbundle of $P^2(M)$ with typical fiber $\tilde{M}_0(n)$: its first prolongation $Q_1(M)$. Thus, an equivalent definition of a conformal structure on M is the following:

A conformal structure on M is the datum of a subbundle $P(M)$ of $P^2(M)$ with structure group $\tilde{M}_0(n)$.

2.9.2.2.1.8 Notation Let $(u^i, u^i_j, u^i_{j\,k})$ be the 2-jet of a map $\varphi : U \to \mathbf{R}^n$, where $U \in \mathcal{V}$ the set of open neighborhoods of 0 in $\mathbf{R}^n$ (cf. 2.9.2.1.1). Let

$$\sum_i (u^i + u^i_j x^j + \frac{1}{2} u^i_{j\,k} x^j x^k) e_i$$

be the polynomial representation of $j^2_0(\varphi)$, where (e_i) is the natural basis of $\mathbf{R}^n$ and $x = \sum x^i e_i$ and $u^i_{j\,k} = u^i_{k\,j}$, as above. One can verify, since $\mathbf{R}^n$ is provided[129] with the classical scalar product, that the elements of $\tilde{M}_1(n)$ are the jets of the form $(0, \delta^i_j, \delta_{jk} a_i - \delta_{ji} a_k - \delta_{ki} a_j)$ with $(a_1, \ldots, a_n) \in \mathbf{R}^n$, and that the first prolongation of the Lie algebra $co(n)$ is the Lie algebra $\tilde{m}_1(n)$ of $\tilde{M}_1(n)$ and thus consisting of jets (a^i_{jk}) such that

$$a^i_{jk} = \delta_{jk} a_i - \delta_{ji} a_k - \delta_{ki} a_j.$$

Finally, the Lie algebra $\tilde{m}_0(n)$ is equal to $co(n) \oplus \tilde{m}_1(n)$.

2.9.2.2.2 Conformal Classical Connections

According to the results given in 2.8.5, we make the following definition:

2.9.2.2.2.1 Definition A conformal classical connection is a Cartan connection for the case $G = \tilde{M}(n)$, the classical Möbius group, and $H = \tilde{M}_0(n)$.

Since the Lie algebra of $\tilde{M}(n)$: $Lie(\tilde{M}(n)) = \tilde{m}(n)$ is $\mathbf{R}^n \oplus co(n) \oplus (\mathbf{R}^n)^*$, a conformal connection ω on P is defined by its components: $(\omega^i, \omega^j_i, \omega_j)$, where[129] $\omega^j_i \in co(n)$.

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[128] S. Kobayashi, *Transformations Groups in Differential Geometry*, Springer, 1972, chapter 1, section 5, pp. 19–23.

[129] $\mathbf{R}^n$ is provided with its canonical standard Euclidean scalar product such that $g_{ij} = \delta_{ij}$ and $\|x\|^2 = g_{ij} x^i x^j$.

The classical Maurer–Cartan structure equations of $\tilde{M}(n)$ are as follows:[130]

$$
\begin{aligned}
d\omega^i &= -\sum_k \omega^i_k \wedge \omega^k, \\
d\omega^i_j &= -\sum_k \omega^i_k \wedge \omega^k_j - \omega^i \wedge \omega_j - \omega_i \wedge \omega^j + \delta^i_k \omega_k \wedge \omega^k, \qquad (1) \\
d\omega_j &= -\sum_k \omega_k \wedge \omega^k_j .
\end{aligned}
$$

If ω denotes a conformal connection on the fiber P with structure group $H = \tilde{M}_0(n)$, the curvature forms (Ω^i_j, Ω_i) and the torsion form (Ω^i)[131] of the connection ω are defined by

$$d\omega^i = -\sum \omega^i_k \wedge \omega^k + \Omega^i,$$

[130] This result is given in S. Kobayashi, *Transformations Groups in Differential Geometry*, op. cit., p. 135. The Maurer–Cartan form ω can be written $\omega = \sum \omega^i e_i + \sum \omega^i_j e^j_i + \sum \omega_j e^j$, where (ω^i_j) is $co(n)$-valued.

[131] *Notes*: We consider P, the bundle of linear frames over M, $\dim M = n$. $G = GL(n, \mathbf{R})$. π denotes the projection $P \to M$. The canonical form θ of P is the $\mathbf{R}^n$-valued 1-form on P defined by $(\theta(X) = u^{-1}(\pi(X))$ for $X \in T_u(P)$, where u is considered as a linear mapping of $\mathbf{R}^n$ onto $T_{\pi(u)}(M)$ [If $u = (X_1, \ldots, X_n)$ is a linear frame at $x = \pi(u)$, u can be given as a linear mapping $u : \mathbf{R}^n \to T_x(M)$ such that $ue_i = X_i$, where $\{e_1, \ldots, e_n\}$ is the natural basis for $\mathbf{R}^n$, $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$, for $i = 1, 2, \ldots, n$. The action of $GL(n, \mathbf{R})$ on P can be interpreted as follows:

Consider $a = (a^i_j) \in GL(n, \mathbf{R})$ as a linear transformation of $\mathbf{R}^n$ that maps e_j into $\sum_i a^i_j e_i$. Then $ua : \mathbf{R}^n \to T_x(M)$ is the composite of the following two mappings:

$$\mathbf{R}^n \xrightarrow{a} \mathbf{R}^n \xrightarrow{u} T_x(M).$$

A connection in the bundle P of linear frames over M is called a linear connection of M. The torsion form Θ of a linear connection Γ is defined as $\Theta = D\theta$, the exterior covariant differential of θ, the canonical form of P.

In the same way, a generalized affine connection of M is defined as a connection in the bundle $A(M)$ of affine frames over M. Now we recall briefly that the torsion tensor field—or simply torsion—T and the curvature tensor field—or simply curvature R—such as T is a tensor field of type (1, 2) and R is a tensor field of type (1, 3) can be expressed in terms of covariant differentiation as follows:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

and

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$

where X, Y, and Z are vector fields on H.

All these classical results can be found, for example, in the book by S. Kobayashi and K. Nomizu, *Foundations of Geometry*, volume I, op. cit.

$$d\omega^i_j = -\sum \omega^i_k \wedge \omega^k_j - \omega^i \wedge \omega_j - \omega_i \wedge \omega^j + \delta^i_j \sum \omega_k \wedge \omega^k + \Omega^i_j, \tag{2}$$
$$d\omega_j = -\sum \omega_k \wedge \omega^k_j + \Omega_j.$$

The Lie algebra $\tilde{m}_0(n)$ of $H = \tilde{M}_0(n)$ is the Lie subalgebra of $\tilde{M}(n)$ defined by $\omega^i = 0$ $(1 \leq i \leq n)$.

Since $\omega(A^*) = A$ for any fundamental vector field generated by an element A of the Lie algebra of H, the restrictions of the forms ω^i vanish on the fibers of P, and the forms (ω^i_j) and (ω_j) generate the cotangent space to these fibers. The restrictions of the forms (ω^i_j) and (ω_j) to the fibers can be identified with the Maurer–Cartan forms of $\tilde{M}_0(n)$ and satisfy the system (1) above with $\omega^i = 0$ for $1 \leq i \leq n$. Thus the relations $\omega^i = 0$ $(1 \leq i \leq n)$ imply that $\Omega^i = 0$, $\Omega^j_i = 0$, $\Omega_j = 0$. Then there exist functions K^i_{jk}, K^i_{jkh}, K_{jkl} on P such that $\Omega^i = \frac{1}{2} K^i_{jk} \omega^j \wedge \omega^k$, $\Omega^i_j = \frac{1}{2} K^i_{jkh} \omega^k \wedge \omega^h$, $\Omega_j = \frac{1}{2} K_{jkh} \omega^k \wedge \omega^h$.

2.9.2.2.2.2 Definition A conformal connection on P is called a normal connection if it is without torsion, i.e., $\Omega^i = 0$, for any $i = 1, 2, \ldots, n$, and if its curvature tensor satisfies the following relation:

$$\sum_i K^i_{jil} = 0. \tag{3}$$

We are going to show that a normal connection is uniquely determined by the datum of the principal bundle P with structure group $\tilde{M}_0(n)$ and of the forms (ω^i, ω^i_j) and that it satisfies the relation

$$\sum_i \Omega^i_i = 0.$$

2.9.2.2.2.3 Theorem *Let P be a subbundle of $P^2(M)$ that defines a conformal structure. Let (ω^i, ω^i_j) be a system of $n + n^2$ differential forms on P such that*

(i) *$\omega^i(A^*) = 0$ and $\omega^i_j(A^*) = A^i_j$ for any fundamental vector field A^* generated by an element (A^i_j, A_i) of the Lie algebra of $\tilde{M}_0(n)$; according to the structure of this Lie algebra $\tilde{M}_0(n)$ given in 2.9.2.2.1.8.*
(ii) *$R^*_a(\omega^i, \omega^i_j) = ad(a^{-1})(\omega^i, \omega^i_j)$ for any $a \in \tilde{M}_0(n)$.*
(iii) *The vertical vectors (i.e., tangent to the fibers) are those that satisfy $\omega^i(X) = 0$, $1 \leq i \leq n$.*
(iv) *$d\omega^i = -\sum \omega^i_k \wedge \omega^k$.*

Then there exists a unique system of forms $(\omega_1, \ldots, \omega_n)$ on P that $(\omega^i, \omega^i_j, \omega_i)$ define a normal connection on P.

Proof. (a) First, let us show that the relations

$$\sum_i K^i_{jil} = 0$$

imply that

$$\sum_i \Omega^i = 0,$$

i.e.,

$$\sum_i K^i_{ijl} = 0. \tag{4}$$

By using exterior differentiation, we get from (iv)

$$\sum_k d\omega^i_k \wedge \omega^k + \sum_{k,j} \omega^i_k \wedge \omega^k_j \wedge \omega^j = 0.$$

Then, using the second relation of the system (2) above, which defines Ω^i_j, and writing that the terms that do not contain the forms ω_i, ω^j_i vanish, we obtain

$$\sum_i \Omega^i_k \wedge \omega^k = 0. \tag{5}$$

Writing explicitly, we get $K^i_{jkl} + K^i_{jlk} + K^i_{lkj} = 0$, whence if $k = i$, according to the relation (4);

$$\sum_i K^i_{jil} = 0,$$

and taking account of the skew symmetry of K^i_{jkl} relative to j, l, we get

$$\sum_i K^i_{ijl} = -\sum_i K^i_{jli} - \sum_i K^i_{lij} = 0.$$

(b) Now *we are going to prove that the required connection is unique*. Let us assume that there exist two systems of forms (ω_i), $(\bar{\omega}_i)$ that define normal connections. Taking account of (i), we have $\bar{\omega}_i(A^*) - \omega_i(A^*) = 0$, for any fundamental vector field, and then $\bar{\omega}_i(X) - \omega_i(X) = 0$ for any vertical vector X, which implies that $\bar{\omega}_i - \omega_i$ is a linear combination of the forms ω^i. We put

$$\bar{\omega}_i - \omega_i = \sum_k A_{ik}\omega^k.$$

Let us denote by Ω^j_i, Ω_i, the curvature forms of the connection $(\omega^i, \omega^j_i, \omega_i)$ and by $\bar{\Omega}^j_i$, $\bar{\Omega}_i$, the curvature forms of the connection $(\omega_i, \omega^j_i, \bar{\omega}_i)$. Taking account of the system (2) above, we have

$$\bar{\Omega}^i_j - \Omega^i_j = \omega^i \wedge (\bar{\omega}_j - \omega_j) + (\bar{\omega}_i - \omega_i) \wedge \omega^j - \delta^j_i \sum (\bar{\omega}_k - \omega_k) \wedge \omega^k,$$

whence we obtain that

$$\sum (\bar{\Omega}^i_i - \Omega^i_i) = -\delta^i_j \sum (\bar{\omega}_k - \omega_k) \wedge \omega^k = -n \sum_i A_{ki}\omega^i \wedge \omega^k.$$

Thus, the conditions $\sum \bar{\Omega}^i_i = \sum \Omega^i_i = 0$ imply the symmetry $A_{ki} = A_{ik}$. We have now

$$\bar{\Omega}^j_i - \Omega^j_i = \omega^i \wedge (\bar{\omega}_j - \omega_j) + (\bar{\omega}_i - \omega_i) \wedge \omega^j = \sum_k (A_{jk}\omega^i \wedge \omega^k + A_{ik}\omega^k \wedge \omega^j).$$

Looking after the skew-symmetric coefficient of $\omega^k \wedge \omega^l$ in $\bar{\Omega}^i_j - \Omega^i_j$, we get that

$$\bar{K}^i_{jkl} - K^i_{jkl} = -\delta^i_l A_{jk} + \delta^i_k A_{jl} + \delta^j_l A_{ik} - \delta^j_k A_{il} \tag{6}$$

and

$$\sum_i (\bar{K}^i_{jil} - K^i_{jil}) = -A_{jl} + nA_{jl} + \delta^j_l \sum A_{ii} - A_{jl} = (n-2)A_{jl} + \delta^j_l \sum A_{ii} = 0,$$

whence we find that $A_{jl} = 0$ if $j \neq l$ and

$$\sum_i A_{ii} = (2-n)A_{jj}$$

for any $j = 1, 2, \ldots, n$, which implies that $n \sum A_{ii} = (2-n) \sum A_{ii}$. Therefore, $\sum A_{ii} = 0$ and, finally, $A_{jj} = 0$ for any j. Finally, the relations (3) and (4) imply that $A_{jl} = 0$ for any j, l.

(c) Now *we prove the existence*. Let $(\mathcal{U}_\alpha, h_\alpha)$ be an atlas of M such that $(\mathcal{U}_\alpha)$ is a locally finite open covering of M and let (f_α) be a partition of unity subordinate to $(\mathcal{U}_\alpha)$. Since $p^{-1}(\mathcal{U}_\alpha)$[132] is diffeomorphic to the product $\mathcal{U}_\alpha \times \tilde{M}_0(n)$, one can build a cross section σ_α over $\mathcal{U}_\alpha$, that is a mapping $\sigma_\alpha : \mathcal{U}_\alpha \to P$ such that $p \circ \sigma_\alpha = \mathrm{Id}$. For any $z \in p^{-1}(\mathcal{U}_\alpha)$ there exists a unique $a \in \tilde{M}_0(n)$ such that $R_{a^{-1}}(z) \in \sigma_\alpha(\mathcal{U}_\alpha)$, and any vector Y tangent to P at z can be uniquely written $Y = R_a(X) + W$, X being tangent to $\sigma_\alpha(\mathcal{U}_\alpha)$ at $R_{a^{-1}}(z)$ and W being a vertical vector, since the tangent space to $\sigma_\alpha(\mathcal{U}_\alpha)$ is a complementary of the tangent space to the fiber.

The vertical vector W is the value at z of a fundamental vector field A^*, where A belongs to $\tilde{m}_0(n)$. Now let us put $\omega_\alpha(X) = (\omega^i(X), \omega^i_j(X), 0)$ and $\omega_\alpha(Y) = ad(a^{-1})\omega_\alpha(X) + A$. Then the form ω defined on P by $\omega(Y) = \sum f_\alpha \omega_\alpha(Y)$ leads to a conformal connection of the form $(\omega^i, \omega^i_j, \omega_j)$. In order to obtain a normal connection, it is sufficient to replace the forms ω_j by the forms

$$\bar{\omega}_j = \omega_j + \sum_k A_{jk}\omega^k$$

with

$$A_{jk} = \frac{1}{2(n-1)(n-2)} \delta^k_j \sum_{i,l} K^i_{lil} - \frac{1}{n-2} \sum_i K^i_{jik},$$

if we denote by K^i_{jkl} the curvature tensor of the connection $(\omega^i, \omega^i_j, \omega_j)$. This fact is a consequence of the relations (3) above with the assumption $\sum \bar{K}^i_{jil} = 0$.

[132] Here, p denotes the projection: $P \to M$.

2.9.2.2.2.4 Properties of Normal Connections

2.9.2.2.2.4.1 Theorem *Let $(\omega^i, \omega^i_j, \omega_j)$ be a normal connection on the fiber bundle P. Then the curvature forms Ω_i, Ω^j_i satisfy the following relations:*

(i) $\sum \Omega^i_j \wedge \omega^j = 0$*, that is* $K^i_{jkl} + K^i_{klj} + K^i_{ljk} = 0$.
(ii) $\sum \omega^i \wedge \Omega_i = 0$*, that is* $K_{jkl} + K_{klj} + K_{ljk} = 0$.

Proof. The relations (i) have already being given (formula (5) above). The formula (ii) can be obtained by exterior differentiation of the last formula of the system (2) taking account of $\sum \Omega^i_i = 0$.

We can therefore deduce the following statement:

2.9.2.2.2.4.2 Theorem *Let $P(M)$ be a conformal structure on M and let (θ^i, θ^i_j) be the restriction to $P(M)$ of the canonical form of $P^2(M)$. Then there exists a unique conformal normal connection $(\omega^i, \omega^i_j, \omega_j)$ on $P(M)$ such that $\omega^i = \theta^i$ and $\omega^i_j = \theta^i_j$. Such a connection satisfies the following relations:*

$$\sum \Omega^i_i = 0$$

and

$$\sum_i K^i_{jil} = 0.$$

The theorem is an immediate consequence of the fact that the forms (θ^i, θ^i_j) satisfy the assumptions of the previous theorem.

2.9.2.2.2.4.3 Extension of the Connection to $P^2(M)$ In order to extend to $P^2(M)$ a conformal normal connection ω, we need to keep in mind that the Lie algebra of $\tilde{M}_0(n)$ is the Lie subalgebra of $G^2(n)$ consisting of elements $(\alpha^i_j, \alpha^i_{jk})$ such that $(\alpha^i_j) \in co(n)$ and (α^i_{jk}) are given by

$$\alpha^j_k = \delta_{jk} a_i - \delta^i_j a_k - \delta^i_k a_j, \tag{7}$$

where $(a_1, \ldots, a_n) \in \mathbf{R}^n$. The extension to $P^2(M)$ of the connection ω, will be a form π with values in $\mathbf{R}^n \oplus g^2(n)$ with components $(\pi^i, \pi^i_j, \pi^i_{jk})$ in the canonical basis of $\mathbf{R}^n \oplus g^2(n)$ such that

(i) the restriction of π to $P(M)$ is of the form $(\omega^i, \omega^i_j, \delta_{jk}\omega_i - \delta^i_j\omega_k - \delta^i_k\omega_j)$,
(ii) for any $a \in G^2(n)$, $R^*_a \pi = ad(a^{-1})\pi$.

The links between conformal connections and riemannian connections will be studied in the exercises.

2.9.2.2.3 Conformal Cartan Connections[133]

Let $\tilde{G}_{n+1}$ be the isotropy subgroup of $P(y_{n+1}) \in M = M_n$, with notation of 1.4 or 2.4, the generalized Möbius space, where P denotes the projection to the projective space, as usual. Since the Möbius group $PO(p+1, q+1)$ acts transitively on the Möbius space M_n and since all the isotropy subgroups are isomorphic one to the other, $M_n = PO(p+1, q+1)/\tilde{G}_{n+1}$.

We assume that $\dim N = \dim M_n$*, i.e.,* $m = n$ *with notation of 2.9.1.3.2.*

2.9.2.2.3.1 Definition Let $\tilde{P}$ be a principal bundle with base N and structure group $\tilde{G}_{n+1}$. By definition, a Cartan connection on $\tilde{P}$ with values in the Lie algebra $\mathcal{L}PO(p+1, q+1) = po(p+1, q+1)$ is called a conformal (generalized) Cartan connection.

We can now give the following results. The proof will be given in the exercises.

2.9.2.2.3.2 Proposition (i) *If there exists a conformal Cartan (generalized) connection on a fiber bundle* $(\tilde{P}, \tilde{\pi}, N, \tilde{G}_{n+1})$*, then there exists a conformal Ehresmann connection on the bundle*

$$\zeta = \tilde{P} \times_{\tilde{G}_{n+1}} M_n = (\bar{M}, \bar{\pi}, N, M_n, PO(p+1, q+1))$$

such that if ∇ *denotes the covariant associated derivative and* $\sigma_{\bar{M}}$ *the canonical section of* $\bar{M}$*,* $\nabla \sigma_{\bar{M}}$ *is a soudure between* N *and* M*.*

(ii) *Conversely, if there exists a conformal Ehresmann connection on a bundle* $\zeta = (\bar{M}, \bar{\pi}, N)$ *with typical fiber* M_n *and structure group* $\tilde{G}_{n+1}$ *such that* $(\nabla \sigma_{\bar{M}})$ *defines a soudure, where* ∇ *stands for the covariant derivative subordinate to the Ehresmann connection and* $\sigma_{\bar{M}}$ *the canonical section of* $\bar{M}$*, then there exists a Cartan connection on the principal bundle associated with* $\bar{M}$*.*

2.9.2.2.3.3 Proposition *Let* $\tilde{P}$ *be a principal bundle with base* N *and structure group* $\tilde{G}_{n+1}$ *and let* $\zeta = (\bar{M}, \bar{\pi}, N)$ *be the bundle with typical fiber* M_n*, associated with* $\tilde{P}$*.*

(i) *If there exists a soudure between* N *and* $\bar{M}$*, then there exists a reduction with structure group* $CO(p, q)$ *of the bundle of frames* $R(N)$ *of the manifold* N*. Such a bundle is called a* $CO(p, q)$ *structure on* N*.*

(ii) *Conversely, if there exists a* $CO(p, q)$ *structure* $Q(N)$ *on* N*, then there exists a soudure between* N *and the bundle* $Q(N) \times_{CO(p,q)} M_n$*, once* $CO(p, q)$ *has been identified with a subgroup of* $\tilde{G}_{n+1}$*.*

133 Most of the results given here have been revealed by J. L. Milhorat, op. cit., starting from the following works: S. Kobayashi, *Transformations Groups in Differential Geometry*, op. cit.; K. Oguie, Theory of conformal connections, *Kodai Math., Sem. Rep.* 19, 1967, pp. 193–224; N. Tanaka, Conformal connections and conformal transformations, *Trans. Amer. Math. Soc.*, 92, 1959, pp. 168–190; A. Toure, Divers aspects des connections conformes, op. cit.

2.9.2.2.3.4 Corollary (i) *If there exists a conformal Cartan connection on the principal bundle* $(\tilde{P}, \tilde{\pi}, N, \tilde{G}_{n+1})$*, then there exists a* $CO(p, q)$ *structure on* N.

(ii) *If there exists a* $CO(p, q)$ *structure* $Q(N)$ *on a paracompact manifold* N*, then there exists a conformal Cartan connection on the principal bundle* $\tilde{Q}(N) = Q(N) \times_{CO(p,q)} \tilde{G}_{n+1}$.

2.10 Conformal Geodesics

2.10.1 Cross Sections and Moving Frames: A Review of Previous Results

2.10.1.1 Classical Results

Let $P(M)$ be a principal bundle with base space M and structure group G. A connection form ω on $P(M)$ is a differential form ω of degree 1 given on $P(M)$ with values in the Lie algebra $\mathcal{L}(G)$ of G such that:

(i) For any $z \in P(M)$, the tangent vectors to $P(M)$ that satisfy $\omega(X) = 0$ constitute a subspace $\mathcal{H}_z$ of $T_z(P(M))$ complementary to V_z the tangent space to the fiber at z.
(ii) For any $a \in G$, if we denote by R_a the right action of a on $P(M)$, we have $R_a^* \omega = ad(a^{-1})\omega$.
(iii) If we denote by A^* the fundamental vector field generated by an element A of G, we have $\omega(A_z^*) = A$, for any $z \in P(M)$.

The datum of a connection form on $P(M)$ is equivalent to what follows: for any $z \in P(M)$ there is a distinguished subspace $\mathcal{H}_z$ of $T_z(P(M))$, called horizontal, transversal to the fibers, such that for any $a \in G$, $\mathcal{H}_{za} = R_a^* \mathcal{H}_z$.[134]

2.10.1.2 Induced Connection in a Local Cross Section

Let U be an open set of M and $s : U \to P(M)$ a local cross section. We define the pullback of the form ω by s or induced connection associated with ω by s as the 1-form defined by

$$s^* \omega = \omega(s'(x)dx).$$

[134] The proof is straightforward. First let w be a connection form on $P(M)$. We get a distribution $\mathcal{H}_z$ that satisfies the required conditions. Conversely, if the distribution $\mathcal{H}_z$ satisfies these conditions, any vector X tangent to $P(M)$ at z can be uniquely written $X = X_h + X_V$, with $X_h \in \mathcal{H}_z$ and $X_V \in V_z$, the tangent space to the fiber at z. Since the action of G is simply transitive on the fibers, there exists a unique element $A \in \mathcal{L}(G)$ such that $A_z^* = X_V$. We put $w(X) = A$, which defines the connection form w. ($\mathcal{L}(G)$ is identified with the space of left-invariant vector fields on G.) For any $A \in \mathcal{L}(G)$, we denote by aA the value at $a \in G$ of the field generated by A. Then, if da is a tangent vector to G at a, we can denote by $a^{-1}da$ the element $A \in \mathcal{L}(G)$ such that $da = aA$.

Let (E^i_j) be a basis of $\mathcal{L}(G)$. We define the forms ω^i_j such that

$$\omega = \sum_{i,j} \omega^i_j E^j_i .$$

Let $(\theta_i)_{1 \le i \le n}$ be the basis of the corresponding cotangent space to M. We put

$$s^* \omega^i_j = \sum_k \Gamma^i_{jk} \theta^k .$$

The functions Γ^i_{jk} are the coefficients of the connection in the local cross section s and the corresponding basis.

2.10.1.3 Passage from One Section to Another

First, we remark that any local section $s : U \to P(M)$ defines a diffeomorphism h from $U \times G$ onto $p^{-1}(U)$ such that for any $(x, g) \in U \times G$, $h(x, g) = s(x)g$. Any other section σ above U is such that $\sigma : U \to P(M), x \to s(x)a(x)$, where $a : U \to G$ is a differentiable mapping. The differential of σ is the sum of two terms:

$*$ the first one obtained by differentiation of the first term above as if $a(x)$ were fixed, which gives $R^*_{a(x)} ds$, if we denote by R_a the right translation $z \to za$.

$*$ the second one, obtained by differentiation of the second term, is the image of $da(x)$, the tangent vector to G at $a(x)$ by the mapping $G \to P(M) : a \to az$ with $z = \sigma(x)$.

Then it is the value at $z = \sigma(x)$ of the fundamental vector field A^* with $A = a^{-1} da$, whence we get

$$\sigma^* \omega = \omega(d\sigma) = \omega(R^*_a ds) + a^{-1} da = ad(a^{-1}) s^* \omega + a^{-1} da,$$

since $\omega(R^*_a ds) = (R^*_a \omega) ds = ad(a^{-1}) \sigma(ds)$, that is,

$$\sigma^* \omega = ad(a^{-1}) s^* \omega + a^{-1} da. \qquad (1)$$

2.10.1.4 Associated Bundles

Let F be a manifold. We assume that G acts differentiably on F, on the left. We define a right action of G on $P(M) \times F$ by the following law:

$$(z, y)a = (za, a^{-1} y) \quad \text{for any } z \in P(M),\ y \in F,\ a \in G.$$

The quotient space $E = P(M) \times F/G$ is a bundle with base M typical fiber F, structure group G. Such a bundle is said to be associated with $P(M)$. We denote by p_E the canonical projection from E onto M. Any point z of $P(M)$ defines a bijective mapping from F onto $p_E^{-1}(x)$, where $x = p(z)$. We associate with any $y \in F$ the class zy consisting of the elements $(za, a^{-1} y)$ of $P(M) \times F$.

2.10.1.5 Parallel Displacement

Horizontal curves of $P(M)$ are curves Γ such that for any $z \in \Gamma$, the corresponding tangent line lies in the horizontal space $\mathcal{H}_z$. Such curves are those $t \to z(t)$ ($t \in I, z(t) \in P(M)$, with I an interval of $\mathbf{R}$) that satisfy

$$\omega(z'(t)dt) = z^*\omega = 0.$$

Horizontal curves of the corresponding associated bundle are curves $t \to z(t)y$, where y is fixed in F and where $t \to z(t)$ is a horizontal path of $P(M)$. We recall the following classical result.

2.10.1.5.1 Theorem *Any differentiable curve $t \to x(t)$ with $t \in [0,1]$ admits a unique horizontal lift in $P(M)$, respectively in E, whose starting point is a given point of $p^{-1}(x(0))$, respectively a given point of $p_E^{-1}(x(0))$.*

2.10.1.6 Moving Frames

Let us assume that the structure group G be a subgroup of a linear group $GL(m,\mathbf{R})$ for a value m not necessarily equal to $n = \dim M$.

2.10.1.6.1 Definition Any $a \in G$ can be identified with the image by a of the canonical basis of $\mathbf{R}^m$. Local sections $s : U \to P(M)$ are called moving frames and denoted by $s : x \to (e_i(x))_{1 \le i \le m}$.

The corresponding local bundle homomorphism associated with s is then

$$U \times G \to P(M) : (x,a) \to s(x)a = \left(\sum_k a_i^k e_k\right)_{1 \le i \le m}.$$

Let us denote by ω_j^i the components of the connection form in the canonical basis of the Lie algebra of $GL(m,\mathbf{R})$. The relation $s^*\omega_j^i = \bar{\omega}_j^i$ can be also written as $ds = s(x)\omega$ or

$$de_i = \sum_k \bar{\omega}_i^k e_k. \tag{2}$$

Let $\sigma : x \to s(x)a(x)$ be another local section. Such a formalism allows us to find again formula (1) above (2.10.1.3). Put $\sigma(x) = (e_i'(x))$. We have

$$e_i'(x) = \sum_{k=1}^{n} a_i^k(x) e_k(x),$$

whence by differentiation,

$$de_i' = \sum_k da_i^k e_k + \sum_k a_i^k de_k = \sum_k \left(da_i^k + \sum_r a_i^r \bar{\omega}_r^k\right) e_k$$
$$= \sum_k \left(da_i^k + \sum_r a_i^r \bar{\omega}_r^k\right) b_k^j e_j',$$

where b_k^j stands for the inverse matrix of a_i^j. Then, the new forms $\bar{\omega}_i^{\prime j} = \sigma^* \omega_i^j$ satisfy

$$\bar{\omega}_i^{\prime j} = \sum_k \left(da_i^k + \sum_r a_i^r \bar{\omega}_r^k \right) b_k^j. \tag{3}$$

We find again formula (1) as

$$\sum_k b_k^j da_i^k = (a^{-1} da)_i^j$$

and

$$\sum_r a_i^r \bar{\omega}_r^k b_k^j = (ad(a^{-1}) s^* \omega)_i^j.$$

The relations (3) can be used to extend the definition of the connection form to the bundle with structure group $GL(m, \mathbf{R})$, obtained by embedding of the structure group.

The parallel displacement of the moving frame (e_i) is defined by the differential system

$$de_i - \sum_k \bar{\omega}_i^k e_k = 0 \quad (1 \le i \le m). \tag{4}$$

Let E be the associated bundle with typical fiber $F = \mathbf{R}^m$. The action of $P(M)$ on F can be denoted by $P(M) \times F \to E$, $(z, y) \to zy$,

$$(e_i, y^i) \to \sum_{i=1}^m y^i e_i.$$

The parallel displacement of the point $zy = \sum y^i e_i$ is then defined by the differential system $d(y^i e_i) = 0$, that is,

$$\sum_i \left(dy^i + \sum_k \bar{\omega}_k^i y^k \right) e_i = 0,$$

that is

$$dy^i + \sum_k \bar{\omega}_k^i y^k = 0 \quad (1 \le i \le m). \tag{5}$$

2.10.2 Conformal Moving Frames

Let $P(M)$ be a conformal structure on M, that is an $\tilde{M}_0(n)$ reduction[135] of the bundle $P^2(M)$; see 2.9.2.2.1.7. Let (θ^i, θ_j^i) be the components of the restriction to $P(M)$ of the canonical form and $\theta = (\theta^i, \theta_j^i, \theta_i)$ the normal connection form;

[135] We recall that $\tilde{M}_0(n)$ is the isotropy group of the origin for the standard Möbius group $\tilde{M}(n)$.

see 2.9.2.2.2.4.2—that satisfies $\theta^i_j = -\theta^j_i$ if $j \neq i$ and $\theta^1_1 = \cdots = \theta^n_n$ (cf. 2.9.2.2.1.7 and 2.9.2.2.1.8).[136] But such a conformal connection is not a connection on $P(M)$, since θ does not take values in the Lie algebra $\tilde{m}_0(n)$. But nevertheless, we can consider θ as a connection on the bundle $P_{\tilde{M}(n)}(M)$ obtained by embedding the structure group $\tilde{M}_0(n)$ into the group $\tilde{M}(n)$, whose elements are said to be "affine 2-frames."

We consider $\tilde{M}(n)$ as the subgroup of $GL(n+2, \mathbf{R})$ consisting of elements that leave the quadratic form q such that

$$q(X) = \sum_{i=1}^{n} (X^i)^2 - 2X^0 X^{n+1}$$

invariant. The connection form θ will be represented by the matrix with values in the Lie algebra $\tilde{m}(n)$,

$$\begin{bmatrix} \tau & \theta_j & 0 \\ \theta^i & \theta^i_j - \tau\delta^i_j & \theta_i \\ 0 & \theta^j & \tau \end{bmatrix},$$

where $\tau = \theta^1_1 = \cdots = \theta^n_n$. Any local section $s : U \to P(M)$ will be represented by an orthonormal moving frame consisting of $(n+2)$ analytic spheres of the Möbius space obtained by "completing" the tangent space $T_x M$, that is, $(A_0, A_1, \ldots, A_{n+1})$ such that (cf. 2.2.1.1)

$$dA_p = \sum_{q=0}^{n+1} \omega^q_p A_q$$

with $\omega^Q_P = s^* \bar{\theta}^Q_P$ with the following conventions:

$$\begin{aligned} \bar{\theta}^i &= \bar{\theta}^{n+1}_i = \theta^i \text{ and } \bar{\theta}^0_i = \bar{\theta}^i_{n+1} = \theta_i, \\ \bar{\theta}^j_i &= \theta^j_i \quad \text{if } i \neq j \text{ and } \bar{\theta}^i_i = 0, \\ \bar{\theta}^0_0 &= -\bar{\theta}^{n+1}_{n+1} = -\theta^i_i. \end{aligned}$$

By assumption, we have

$$A_0^2 = A_{n+1}^2 = 0, \quad A_0.A_{n+1} = -1, \quad A_0.A_i = A_{n+1}.A_i = 0, \quad A_i.A_j = \delta^i_j, \tag{1}$$

for $1 \leq i, j \leq n$. If

$$A = \sum_{p=0}^{n+1} y^p A_p$$

136 This result comes from the fact that with the standard canonical scalar product the Lie algebra $co(n)$ consists of matrices (a^j_i) such that $a^j_i + a^i_j = 0$ if $j \neq i$ and that a^i_i is independent of i.

is a point of the corresponding bundle, we put

$$A.A = \sum_{i=1}^{n} y_i^2 - 2y^0 y^{n+1}.$$

2.10.2.1 Cartan's Theory[137]

E. Cartan introduces a moving frame A_p such that the relations (1) are satisfied. Then, he defines

$$dA_p = \sum_{q=0}^{n+1} \omega_p^q A_q$$

$$\omega_0^{n+1} = \omega_{n+1}^0 = 0;\ \omega_0^0 + \omega_{n+1}^{n+1} = 0;\ \omega_0^i = \omega_i^{n+1};\ \omega_{n+1}^i = \omega_i^0;\ \omega_i^j + \omega_j^i = 0 \quad (2)$$

(formulas obtained by differentiation of previous formulas (1)). E. Cartan assumes that M is endowed with a riemannian metric g, and introduces a moving co–frame ω^i such that the metric $\sum(\omega^i)^2$ is conformal to g, that is that the dual frame e_i consists of vectors of the same norm, orthogonal to each other. He assumes then that $dA_0 = \sum_{i=1}^{n} \omega_i A_i$, whence $\omega_0^0 = \omega_{n+1}^{n+1} = 0$ and $\omega_0^i = \omega^i$, which permits him to identify the sphere-point A_0 with the origin of the affine moving frame $(x, e_i(x))$. Then, he assumes that the connection (ω_i^j) is without torsion, that is,

$$d\omega_0^p = \sum_q \omega_0^q \wedge \omega_p^q.$$

And then for $p = 0$,

$$\sum_j \omega^j \wedge \omega_0^j = 0,$$

and for $p = i$,

$$d\omega^i = \sum_{j=1}^{n} \omega^j \wedge \omega_j^i.$$

The formulas $\omega_j^i + \omega_i^j = 0$ and

$$d\theta^i = \sum_{j=1}^{n} \omega^j \wedge \omega_i^j$$

show that the forms ω_j^i are equal to the forms of the riemannian connection associated with the quadratic form $\sum(\omega^i)^2$. Moreover, the relations

$$\sum_j \omega^j \wedge \omega_0^j = 0$$

[137] E. Cartan, Les espaces à connexions conformes, *Annales de la Société polonaise de Maths.*, 2, 1923, pp. 171–221. E. Cartan used the quadratic form q' such that $q'(X) = \sum(X^i)^2 + 2X^0 X^{n+1}$, which leads to replace A_{n+1} by $-A_{n+1}$.

show that if we put

$$\omega_j = \omega_j^0 = \sum_{k=1}^{n} \pi_{jk}\omega^k,$$

we have $\pi_{jk} = \pi_{kj}$.

Finally, E. Cartan shows that one can determine the symmetric coefficients π_{jk} in order that the curvature tensor K^j_{ikh} defined by $\Omega_i^j = K^j_{ikh}\omega^h \wedge \omega^k$,

$$d\omega_j^i = -\sum \omega_p^i \wedge \omega_j^p + \delta_j^i \sum_k \omega_k \wedge \omega^k + \Omega_j^i$$

satisfies

$$\sum_j K^j_{ijh} = 0.$$

Then he shows that such a condition determines uniquely the π_{jk}. This connection is called a normal connection. It is the connection defined in 2.9.2.2.2.2. The consistency of E. Cartan's presentation with the previous one will be studied in the exercises.

2.10.3 The Theory of Yano

K. Yano starts from conformal moving frames not necessarily orthonormal, associated with the standard frame $(\delta/\partial x^i)$ subordinate to a system of coordinates. The tangent space $T_x(M)$ is completed by a point at infinity not fixed beforehand and becomes a Möbius space. The "A_p" are the "analytic spheres" of such a space such that $A_0(x)$ stands for the "point-sphere" x. The fundamental quadratic form is now $g_{ij}X^iX^j - 2X^0X^{n+1}$ with the following conditions:[138]

$$\begin{cases} A_0^2 = A_\infty^2 = 0, \ A_0 \cdot A_\infty = 1, \\ A_i \cdot A_j = g_{ij}, \quad \text{where } g_{ij} \text{ denotes the "metric tensor"} \\ \qquad\qquad\qquad \text{of the riemannian manifold } M. \end{cases} \tag{1}$$

By differentiation of these relations and putting $dA_p = \sum \omega_p^q A_q$, one obtains

$$\begin{cases} \omega_i^\infty = \sum g_{ij}\omega_0^j, & \omega_i^0 = \sum_j g_{ij}\omega_\infty^j, \\ \omega_0^\infty = 0, & \omega_\infty^0 = 0, \omega_0^0 + \omega_\infty^\infty = 0, \\ dg_{ij} = \sum_{k,h} g_{ik}\omega_j^k + g_{hj}\omega_i^h. \end{cases} \tag{2}$$

Then one assumes that $\omega_0^i = dx^i$ and then

$$dA_0 = \omega_0^0 A_0 + \sum_{i=1}^{n} A_i dx^i.$$

[138] K. Yano, Sur les circonférences généralisées dans les espaces à connexion conforme, *Proc. Imp. Acad. Tokyo*, 14, 1938, pp. 329–332.

The moving frames that satisfy such conditions are called seminatural moving frames.

Conditions (2) imply that $d(ln\sqrt{|g|}) = \sum \omega_i^i$, where g denotes the classical $g = \det(g_{ij})$. We can thus deduce that there exists a conformal change of the metric tensor g, $g \to e^{2\lambda} g$, such that the components $\bar{\omega}_i^j$ corresponding to the new metric tensor $\bar{g}$, subordinate to the same moving frame A_p, satisfy $\sum_i \bar{\omega}_i^i = 0$. We can effectively determine the connection by the following assumptions:

(i) The connection is without torsion, that is, satisfying

$$\sum_k \omega^k \wedge \omega_k^i = 0,$$

which is equivalent to $\Gamma_{kh}^i = \Gamma_{hk}^i$ if we put

$$\omega_k^i = \sum \Gamma_{kh}^i dx^h.$$

(ii) The conformal curvature tensor defined by $\Omega_i^j = K_{ikh}^j dx^k \wedge dx^h$ with $\Omega_i^j = d\omega_i^j + \sum \omega_p^j \wedge \omega_i^p - \delta_i^j \sum_k \omega_k \wedge \omega^k$ satisfies $\sum_j K_{ijh}^j = 0$.

2.10.4 Conformal Normal Frames Associated with a Curve

Subsequently, we assume that the manifold M is endowed with a class $(\mathcal{C})$ of riemannian conformal metric tensors. Let $\gamma : I \to M, t \to x(t)$ be a curve in M, where I denotes an interval of $\mathbf{R}$. A lift of γ in $P(M)$ is a mapping $\sigma : I \to P(M)$ such that $p \circ \sigma(t) = x(t)$ for any t in I, where $p : P(M) \to M$ denotes the standard canonical projection.

The mapping σ can be represented by a conformal moving frame (B_p) in the same way as the local sections of $P(M)$ (see 2.10.2 above and the exercises below). Such a frame will be called a normal moving frame subordinate to γ if it satisfies

$$\sum_{i=1}^{n} \sigma^* \theta_i^i = 0,$$

and then $\sigma^* \theta_i^i = 0$ for any i. Let $s = (A_p)$ be a normal moving frame defined on an open set U of M containing γ. One obtains such a frame (B_p) by putting

$$B_p(t) = A_p(x(t)). \tag{1}$$

Conversely, if the interval I is compact, and if the curve γ is a regular simple one and if (B_p) is a normal frame subordinate to γ, one can prove that there exists a normal moving frame (A_p) defined over an open set U of M containing γ that satisfies the relation (1).

Any normal frame $\sigma = (B_p)$ subordinate to γ satisfies the following differential system:

$$\begin{cases} \dfrac{dB_0}{dt} = \sum_{i=1}^{n} \pi^i B_i, & \dfrac{dB_{n+1}}{dt} = \sum_{i=1}^{n} \pi_i B_i, \\ \dfrac{dB_i}{dt} = \pi_i B_0 + \sum_{k=1}^{n} \pi_i^k B_k + \pi^i B_{n+1}, & \end{cases} \tag{2}$$

where we put $\sigma^*\theta^i = \pi^i dt, \sigma^*\theta_i = \pi_i dt, \sigma^*\theta_i^k = \pi_i^k dt$, that is, also $\pi^i = \theta^i(\sigma'(t)), \pi_i = \theta_i(\sigma'(t))$, and $\pi_i^k = \theta_i^k(\sigma'(t))$. The study of the changes of frames will be made below in the exercises.

2.10.5 Conformal Geodesics

2.10.5.1 Basic Fields

Let us consider $\theta = (\theta^i, \theta_j^i, \theta_i)$ the normal connection form on the bundle $P(M)$. For any $\xi \in \mathbf{R}^n$, there exists a unique vector field $B(\xi) : z \to B_z(\xi)$ on $P(M)$ such that

(i) $\theta^i(B(\xi)) = \xi^i$, for any $i = 1, 2, \ldots, n$,
(ii) $\theta_j^i(B(\xi)) = \theta_i(B(\xi)) = 0$, for $i, j = 1, 2, \ldots, n$.

Condition (ii) means that $B(\xi)$ is horizontal.

2.10.5.1.1 Definition (I) The fields $B(\xi)$ that satisfy the conditions (i) and (ii) above with $\xi \neq 0$ are called standard horizontal fields or basic fields.

(II) The conformal geodesics of M are the projections on M of integral curves of basic fields.

It is convenient to start from the above definition in order to identify conformal geodesics according to such a definition with the conformal circles of E. Cartan and K. Yano and with the "conformal null curves" of A. Fialkow[139] without comparing the differential equations of the curves as made by K. Oguie.[140]

[139] (a) E. Cartan, Les espaces à connexions conformes, *Annales de la Société polonaise de Maths.*, 2, 1923, pp. 171–221. (b) K. Yano, α) Sur la théorie des espaces à connexion conformes, *Journal of Faculty of Sciences, Imperial University of Tokyo*, vol. 4, 1939, pp. 40–57. (β) Sur les circonférences généralisées dans les espaces à connexion conforme, *Proc. Imp. Acad. Tokyo*, 14, 1938, pp. 329–332. (γ) see also: The theory of Lie derivatives and its applications, North-Holland, 1957, Chapter VII, pp. 158–160. (c) A. Fialkow, The conformal theory of curves, *Ann. Math. Soc. Trans.*, 51, 1942, pp. 435–501.

[140] K. Ogiue, Theory of conformal connection, *Kodai. Maths. Sem. Rep.*, 19, 1967, pp. 193–224.

2.10.5.2 Conformal Moving Frames Associated with a Conformal Geodesic

Let $\gamma : I \to M$ be a conformal geodesic of M. According to the above definition, there exists a lift of γ in $P(M), \sigma : t \to \sigma(t)$ such that $\sigma^*\theta^i = \xi^i = c, \sigma^*\theta^i_j = 0, \sigma^*\theta_i = 0$. In terms of moving frames, one can give the equivalent following statement:

There exists a normal frame associated with $\gamma, \sigma : t \to (B_0(t), \dots, B_{n+1}(t))$ such that

$$\frac{dB_0}{dt} = \sum_{i=0}^{n} \xi^i B_i, \quad \frac{dB_i}{dt} = \xi^i B_{n+1}, \quad \frac{dB_{n+1}}{dt} = 0. \tag{1}$$

The corresponding lift of γ in the bundle $P^1(M)$ satisfies the following differential system:

$$x'(t) = \sum_{i=1}^{n} \xi^i e_i, \quad \frac{de_i}{dt} = 0, \qquad 1 \le i \le n. \tag{2}$$

Therefore, we can deduce that γ is a geodesic for a riemannian structure of the class $\mathcal{C}$. The fact that the forms $\sigma^*\theta_i = \theta_i(x'(t)dt)$ vanish shows that $x'(t)$ is an eigenvector of the corresponding Ricci tensor (cf. exercises below). Therefore, conformal geodesics are Fialkow conformal null curves (cf. footnote 139c). The converse will be studied later.

2.10.5.3 Generalized Yano Circles

Let us start from a normal frame (B_p) that satisfies condition (1) above. We put

$$\mathcal{C}_0 = B_0 - t\sum \xi^i B_i + \frac{t^2}{2}\sum \xi_i^2 B_{n+1},$$
$$\mathcal{C}_i = B_i - t\sum \xi^i B_{n+1}, \quad \mathcal{C}_{n+1} = B_{n+1}.$$

We obtain a normal frame associated with γ such that $d\mathcal{C}_p/dt = 0$ for any $p = 0, 1, \dots, n+1$.

One can give the following interpretation of conditions (1): the frame $(B_p(t))$ is deduced from a fixed frame $(\mathcal{C}_p)$ by the translation

$$\begin{cases} B_0 = \mathcal{C}_0 + t\sum \xi^i \mathcal{C}_i + \dfrac{t^2}{2}\sum \xi_i^2 \mathcal{C}_{n+1}, \\ B_i = \mathcal{C}_i + t\xi^i \mathcal{C}_{n+1}, \qquad B_{n+1} = \mathcal{C}_{n+1}. \end{cases} \tag{3}$$

The relative trajectory of the point $B_0(t)$ in the frame $(\mathcal{C}_p)$ is defined by the first relation of the system (3). Its canonical projection on the Möbius space is a circle or a line. In particular, if $M = \mathbf{R}^n$, conformal geodesics of $\mathbf{R}^n$ are circles or lines.

In the general case, it appears that conformal geodesics are generalized Yano circles.[141] The converse results from the following theorem, which will be proved below in the exercises.

2.10.5.3.1 Theorem *Let $\gamma : I \to M$ be a curve and (A_p) a normal frame subordinate to γ. Then γ is a conformal geodesic of M if and only if there exists a function ρ that does not vanish on I and such that $(d^3/dt^3)(\rho A_0) = 0$.*

According to this theorem there is identity between the notions of conformal geodesic and of generalized circle as already proved by K. Ogiue[142] by comparing the corresponding differential equations of these curves.

2.10.5.3.1 The Elie Cartan's definition

First, we notice that the relations (1) above can be simplified by the choice of a fixed orthogonal matrix a_i^j such that $a_i^j = \frac{\xi^j}{\xi}$, for any j, where $\xi = \left(\sum (\xi^i)^2\right)^{\frac{1}{2}}$ and by the consideration of the normal frame $\bar{B}_p$ subordinate to γ defined by $\bar{B}_0 = \frac{1}{\xi} B_0$, $\bar{B}_i = \sum a_i^j B_j$, $\bar{B}_{n+1} = \xi B_{n+1}$. This frame satisfies the following relations:

$$\frac{d\bar{B}_0}{dt} = \bar{B}_1, \ \frac{d\bar{B}_1}{dt} = \bar{B}_{n+1}, \ \text{ and } \ \frac{d\bar{B}_p}{dt} = 0, \ \text{ for } \ p \geq 2. \tag{4}$$

Moreover, we can notice that a nonlinear change of variable cannot in general conserve the conditions (1) and the property of γ to be a conformal geodesic.

We are now going to transform the relations (4) by a change of parameter and a change of frame not necessarily normal and we will get a characterization of conformal geodesics in any parametrization.

By a change of the parameter $t = t(u)$ such that $\frac{dt}{du} = \pi$, the system (4) above can be written:

$$\frac{d\bar{B}_0}{du} = \pi \bar{B}_1, \ \frac{d\bar{B}_1}{du} = \pi \bar{B}_{n+1}, \ \frac{d\bar{B}_{n+1}}{du} = 0. \tag{5}$$

Then we use the following change of orthonormal conformal frame:

$$A_0 = \rho \bar{B}_0, \ A_1 = \bar{B}_1 + \rho a \bar{B}_0, \ A_i = \bar{B}_i \ \text{ for } 2 \leq i \leq n, \tag{6}$$

$$A_{n+1} = \frac{1}{\rho} \bar{B}_{n+1} + a \bar{B}_1 + \frac{1}{2} \rho a^2 B_0,$$

with $\rho = \frac{1}{\pi}$ and where a is any function. We obtain:

$$\begin{cases} \frac{dA_0}{du} = \pi_0 A_0 + A_1 & \frac{dA_1}{du} = \pi_1 A_0 + A_{n+1} \\ \frac{dA_{n+1}}{du} = \pi_1 A_1 - \pi_0 A_{n+1} \ \text{ and } & \frac{dA_i}{du} = 0 \ \text{ if } 2 \leq i \leq n. \end{cases} \tag{7}$$

141 K. Yano, Sur les circonférences généralisées dans les espaces à connexion conforme, *Proc. Imp. Acad. Tokyo*, 14, 1938, pp. 329–332.

142 K. Ogiue, Theory of conformal connection, *Kodai. Maths. Sem. Rep.*, 19, 1967, pp. 193–224.

with

$$(7\,bis)\pi_0 = \frac{\rho'}{\rho} - a,\ \pi_1 = a\frac{\rho'}{\rho} + a' - \frac{a^2}{2}.$$

That is the moving nonnormal frame considered by E. Cartan.[143]

Conversely, if A_p is a moving conformal orthonormal frame which satisfies a differential system of the form (7), one can determine two functions a, ρ which satisfy the differential system (7 bis). The relations (6) determine, then, a moving orthonormal frame (B_p) which satisfy (5). Then, we have obtained:

2.10.5.4 Theorem *There exists a change of parameter that allows us to transform a curve γ into a conformal geodesic if and only if we can subordinate to it a moving conformal frame (A_p) that satisfies a differential system of the form (7).*

Then there is identity between the notion of conformal geodesic in any parametrization and that of E. Cartan's "conformal circles."

2.10.5.5 Fialkow's Definition[144]

Theorem 2.10.5.4 allows us to study the identification of the conformal null curves (cf. 2.10.5.2 above).

2.10.5.5.1 Theorem *There exists a change of parameter that allows the transformation of a curve γ into a conformal geodesic if and only if γ is a Fialkow's "conformal null curve," that is to say, if and only if there exists a "riemannian metric" of the class $\mathcal{C}$ such that γ is a corresponding geodesic and that the tangent vector to γ is at any point an eigenvector of the Ricci tensor.*

The proof will be given below in the exercises.

2.11 Generalized Conformal Connections[145]

2.11.1 Conformal Development

2.11.1.1 Definition Let (M, π, N) be a fiber bundle and let $H(M)$ be an Ehresmann connection on M. Let $t \to z_t$ be a differentiable curve on M and $\gamma : t \to \gamma_t = \pi(z_t)$ its projection onto N. The differentiable curve (u_t) defined in the fiber $\pi^{-1}(\gamma_{t_0})$ over a point γ_{t_0} of (γ_t) by $u_t = \tau_\gamma^{-1}(z_t)$, where τ_γ is the parallel displacement of $\pi^{-1}(\gamma_{t_0})$ onto $\pi^{-1}(\gamma_t)$ subordinate to γ, is called the development of the curve z_t in the fiber $\pi^{-1}(\gamma_{t_0})$.

[143] E. Cartan, Les espaces à connexions conformes, Annales de la Société polonaise de Maths., 2, 1923, pp. 171–221.

[144] A. Fialkow, The conformal theory of curves, *Ann. Math. Soc. Trans.*, 51, 1942, pp. 435–501.

[145] These results are due to J. L. Milhorat, Sur les connections conformes, Thesis, 1985, Université Paul Sabatier, Toulouse.

2.11.1.2 Special Case

Let (M, π, N, F, G) be a fiber bundle with typical fiber F and structure group G and let $H(M)$ be an Ehresmann G-connection on this bundle. We consider a differentiable curve $t \to z_t$ of M and $\gamma : t \to \gamma_t$ its projection on N and (u_t) the development of z_t in the fiber $\pi^{-1}(\gamma_{t_0})$ over a point γ_{t_0} of (γ_t). We have $z_t = \tau_\gamma(u_t)$, where τ_γ is the parallel displacement in M subordinate to γ.

If we denote by (p_t) the horizontal lift of γ in the principal bundle P associated with M, with origin p_0, a point belonging to the fiber over γ_{t_0}, according to 2.8.8.4, we have

$$u_t = \tau_\gamma^{-1}(z_t) = p_0 \circ p_t^{-1}(z_t). \tag{a}$$

We can notice that p_0^{-1} allows us to identify the development of (z_t) and the curve y_t of F defined by

$$y_t = p_t^{-1}(z_t) \tag{b}$$

(but such a curve (y_t) obviously depends of the choice of p_0).

2.11.1.3 Definition Let P be a principal bundle with base N, structure group H, where H is a closed subgroup of a Lie Group G such that $\dim G/H = \dim N$, and let w be a Cartan connection on P. According to 2.8.5 and 2.8.6 above, w defines an Ehresmann connection on the fiber bundle X with typical fiber G/H subordinate to P. Let σ_X be the canonical section of this bundle X and let $\gamma : t \to \gamma_t$ be a differentiable curve in N. The development in the fiber of X over a point of (γ_t) of the differentiable curve $(\sigma_X \circ X)$ of X is called by definition the development of the differentiable curve γ of N in the fiber of X over a point of (γ_t).[146]

2.11.1.4 Study of the Conformal Case

Let $\tilde{P}$ be a principal bundle on an n-dimensional manifold N with $n > 2$. We denote by $\tilde{\pi}$ the projection and the structure group by $\tilde{G}_{n+1}$, where according to 2.9.1.3, $\tilde{G}_{n+1}$ is the isotropy subgroup of the point of the Möbius space $M_n : P(y_{n+1}) = P((e_0 - e_{n+1})/2)$[147] with the notation of 2.9.1.3 and 2.9.2.2.3. Let $\tilde{w}$ be a Cartan conformal connection on $\tilde{P}$. According to 2.9.2.2.3.2, we know that $\tilde{w}$ defines an Ehresmann conformal connection $H(\bar{M})$ on the fiber bundle $\zeta = (\bar{M}, \bar{\pi}, N)$ with typical fiber the Möbius space $M = M_n$, subordinate to $\tilde{P}$.

More precisely, let us consider the principal bundle

$$\bar{P} = \tilde{P} \times_{\tilde{G}(n+1)} PO(p+1, q+1),$$

ζ can be identified with the fiber bundle with typical fiber M_n subordinate to $\bar{P}$, and $H(\bar{M})$ is the connection on ζ associated with the principal connection with form $\bar{w}$ on

[146] J. Dieudonné, *Elements d'Analyse*, tome 4, Gauthier-Villars, Paris, 1971.

[147] P denotes the projection from $E_n(p, q)$ onto $P(E_n(p, q))$, the corresponding projective space.

$\bar{P}$ such that $i_1^* \bar{w} = \tilde{w}$, where i_1 denotes the injective morphism of the principal bundle from $\tilde{P}$ into $\bar{P}$ defined by $i_1(\tilde{p}) = \overline{(\tilde{p}, e)}$, where $e = P(\mathrm{Id}(E_{n+2}(p+1, q+1)))$ and $\overline{(\tilde{p}, e)}$ denotes the class of $(\tilde{p}, e)$. Abusively, for any $\tilde{p} \in \tilde{P}$, respectively $\bar{p} \in \bar{P}$, that are projected onto $x \in N$, we will denote by the same letter, $\tilde{p}$, respectively $\bar{p}$, the diffeomorphism of M_n on the fiber of ζ over x defined by $\tilde{p}(\bar{z}) = \tilde{p} \bullet \bar{z}$, respectively $\bar{p}(\bar{z}) = \bar{p} \bullet \bar{z}$, for any $\bar{z} \in M_n$, where $\tilde{p} \bullet \bar{z}$, respectively $\bar{p} \bullet \bar{z}$, denotes the corresponding class modulo $\tilde{G}_{n+1}$ of $(\tilde{p}, \bar{z}) \in \tilde{P} \times M_n$, respectively $(\bar{p}, \bar{z}) \in \bar{P} \times M_n$.

The canonical section $\sigma_{\bar{M}}$ of the bundle ζ, according to 2.9.2.2.3, is defined by $\sigma_{\bar{M}} = \tilde{p}(\bar{y}_{n+1}) = \tilde{p} \bullet \bar{y}_{n+1}$ for $x \in N$, $\tilde{p} \in \tilde{P}_x$, and where $\bar{y}_{n+1} = P(y_{n+1}) = P((e_0 - e_{n+1})/2)$ with the above notation.

2.11.1.4.1 Definition Let $\gamma : t \to \gamma_t$ be a differentiable curve of N. According to Definition 2.11.1.3 above, the development of the path $(\sigma_{\bar{M}} \circ \gamma)$ in the fiber over a point of (γ_t) is called the conformal development of the path γ of N.

Let $\gamma : t \to \gamma_t$ be a path of N. Let $\bar{p}_0$ be an element of the fiber of $\bar{P}$ over a point γ_{t_0} and let $(\bar{p}_t)$ be the horizontal lift of the path γ in $\bar{P}$ with origin $\bar{p}_0$. There exists a path in $PO(p+1, q+1)$ such that

$$\bar{p}_t \bullet g_t = i_1(\tilde{p}_t), \tag{c}$$

where $(\tilde{p}_t)$ is a path in $\tilde{P}$. According to 2.11.1.2 (formula (a)), the conformal development (u_t) of the path γ in the fiber $\bar{M}_{\gamma_{t_0}}$ satisfies

$$\bar{u}_t = \bar{p}_0 \circ \bar{p}_t^{-1}(\sigma_{\bar{M}}(\gamma_t)) = \bar{p}_0 \circ g_t \circ (i_1(\tilde{p}_t))^{-1}(i_1(\tilde{p}_t)(\bar{y}_{n+1})),$$

where abusively $\tilde{p}_t(\bar{y}_{n+1})$ is identified with $i_1(\tilde{p}_t)(\bar{y}_{n+1})$. Thus

$$u_t = \bar{p}_0(g_t \bullet \bar{y}_{n+1}). \tag{c$_1$}$$

Using the derivative of (c), we obtain that

$$\dot{\bar{p}}_t \bullet g_t + \bar{p}_t \bullet \dot{g}_t = (i_1)_*(\dot{\tilde{p}}_t),$$

(We recall that given a mapping f of a manifold M into another manifold M', the differential at p of f is the linear mapping f_* of $T_p(M)$ into $T_{f(p)}(M)$ defined as follows. For each $X \in T_p(M)$, choose a path $x(t)$ in M, such that X is tangent to $x(t)$ at $p = x(t_0)$. Then $f_*(X)$ is the vector tangent to the path $f(x(t))$ at $f(p) = f(x(t_0))$. Cf. for example, S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, vol. 1, op. cit., p. 8.)

Whence

$$\bar{w}(\dot{\bar{p}}_t \bullet g_t + \bar{p}_t \bullet \dot{g}_t) = g_t^{-1} \bullet \dot{g}_t = \bar{w}((i_1)_*(\dot{\tilde{p}}_t)) = \tilde{w}(\dot{\tilde{p}}_t),$$

and therefore,

$$g_t^{-1} \bullet \dot{g}_t = \tilde{w}(\dot{\bar{p}}_t). \tag{d}$$

2.11.1.4.2 Definition Let a be an element of $\mathbf{R}^n$. We call by definition the basic field on $\tilde{P}$ the vector field H_a on $\tilde{P}$ defined by $\tilde{w}(H_a) = a$.

The existence and uniqueness of the vector field H_a come from the fact that by definition, $\tilde{w}_{\tilde{p}}$ is a bijective map from $T_{\tilde{p}}(\tilde{P})$ onto the Lie algebra $\mathcal{L}PO(p+1, q+1) = po(p+1, q+1)$ isomorphic to the Lie algebra $\mathbf{R}^n \oplus co(p, q) \oplus (\mathbf{R}^n)^*$, for any $\tilde{p} \in \tilde{P}$. Thus H_{e_i}, with (e_i) a basis of $\mathbf{R}^n$, and A^* with A in the Lie algebra $(\mathbf{R}^n)^* \oplus co(p, q)$, define a parallelism on $\tilde{P}$.

In the paper given in footnotes 140 and 142 in section 2.10.5, as a reference, K. Oguie calls the projections onto N of the integral curves of basic fields on $\tilde{P}$ conformal geodesics of N.

Let $t \to \gamma_t$ be a conformal geodesic on N. By definition, there exists a path $t \to \varphi_t$ in $\tilde{P}$ such that

$$\begin{cases} \tilde{\pi}(\varphi_t) = \gamma_t, \\ \tilde{w}(\dot{\varphi}_t) = a, a \in \mathbf{R}^n. \end{cases}$$

Let $(\bar{p}_t)$ be the horizontal lift of the path (γ_t) in $\bar{P}$ with origin $i_1(\varphi_{t_0})$. There exists a path (g_t) in $PO(p+1, q+1)$ such that $\bar{p}_t \bullet g_t = i_1(\varphi_t)$ and $g_t = e$ (where e denotes the identity element in $PO(p+1, q+1)$).

According to 2.11.1.4.1, formulas (c_1) and (d), the conformal development of the conformal geodesic (γ_t) in the fiber $\bar{M}_{\gamma_{t_0}}$ is the path (u_t) defined by

$$u_t = \bar{p}_0(g_t \bullet \bar{y}_{n+1}), \tag{e}$$

with

$$g_t^{-1} \bullet \dot{g}_t = \tilde{w}(\dot{\varphi}_t) = a, \tag{f}$$

where there, $a \in \mathbf{R}^n$ is identified with an element of the Lie algebra

$$\mathcal{L}PO(p+1, q+1) = po(p+1, q+1).$$

2.11.1.4.3 Definition (α) By definition we call a "conformal circle" of the Möbius space M_n any subset $M_n : M_n \cap P(H)$, where P is the classical projection from the space onto its projective associated space and where $H = P_1 \oplus \mathrm{Vect}\{a\}$, with P_1 a hyperbolic plane in $E_{n+2}(p+1, q+1)$ and a a nonisotropic element in $E_{n+2}(p+1, q+1)$, $a \notin P_1$.

(β) Then by definition, we call the projective subspace $P(F)$, where F is a totally isotropic plane in $E_{n+2}(p+1, q+1)$, a "minimal line" in M_n.

2.11.1.5 Study of a "Conformal Circle"

Let $\tilde{C} = M_n \cap P(H)$ be a "conformal circle," with $H = P_1 \oplus \mathrm{Vect}\{a\}$, where $a^2 \neq 0$ in $E_{n+2}(p+1, q+1)$. We can assume that $a^2 = \varepsilon = \pm 1$. Let $\{\varepsilon_0, \varepsilon_{n+1}\}$ be a standard basis of P_1 with $(\varepsilon_0)^2 = 1$, $(\varepsilon_{n+1})^2 = -1$. As usual, one can construct an orthonormal basis $\{\varepsilon_i\}$, $i = 1, 2, \ldots, n$, of $P_1^{\perp}$ such that $\varepsilon_1 = a$.

2.11.1.5.1 Lemma *Let P be the classical projection from E_{n+2}, onto $P(E_{n+2})$, its projective associated space. Then $\tilde{C} = P((S^p \times S^q) \cap H)$.*

First, we note that according to a classical result,[148] $P(S^{n+1}) = P(E_{n+2})$, where S^{n+1} is the unit sphere in E_{n+2}. Then M_n is homeomorphic to $\frac{S^p \times S^q}{\mathbf{Z}_2}$. We consider $E_{n+2}(p+1, q+1)$ as the product of the Euclidean spaces $\mathbf{R}^{p+1}$ and $\mathbf{R}^{q+1}$. The corresponding quadratic form q defined on E_{n+2} can be written as $q(x, y) = \|x\|^2 - \|y\|^2$, for any $x \in \mathbf{R}^{p+1}$, any $y \in \mathbf{R}^{q+1}$. Then M_n appears as the image by P of the set of $(x, y) \in \mathbf{R}^{p+1} \times \mathbf{R}^{q+1} \setminus \{(0, 0\}$ such that $\|x\|^2 - \|y\|^2 = 0$ and $\|x\|^2 + \|y\|^2 = 1$, and, then of the product of the corresponding spheres with radius $\frac{1}{\sqrt{2}}$ of $\mathbf{R}^{p+1}$ and $\mathbf{R}^{q+1}$.

Moreover, if $P((x, y)) = P((x', y'))$, then $x' = kx$ and $y' = ky$ with $\|x'\| = \|y'\| = \|x\| = \|y\| = 1/\sqrt{2}$, then $k = \pm 1$. Then $P((S^p \times S^q) \cap H) = P((S^p \times S^q) \cap (H_*))$, where $H_* = H \setminus \{0\}$, whence we find that $\tilde{C} = P((S^p \times S^q) \cap H)$. If $a^2 = \varepsilon_1^2 = 1$, then $\tilde{C} = P(C \times \{-\varepsilon_{n+1}, \varepsilon_{n+1}\})$, where C is the circle in E_{n+2} defined as $C = S^p \cap \mathrm{Vect}\{\varepsilon_0, \varepsilon_1\}$. If $a^2 = \varepsilon_1^2 = -1$, then $\tilde{C} = P(\{-\varepsilon_0, \varepsilon_0\} \times C')$, where C' is the circle in E_{n+2} defined as $C' = S^q \cap \mathrm{Vect}\{\varepsilon_1, \varepsilon_{n+1}\}$. When E_n is an Euclidean space, M_n can be identified with S^n, and the corresponding circles are the usual "big circles" of S^n.

2.11.1.5.2 Equivalence of Such a Definition of a Conformal Circle with the Analytic Definition Given in the Euclidean Case by E. Cartan[149]

Proof. First we prove that the previous definition implies that of Elie Cartan. Put $x_0' = (\varepsilon_0 + \varepsilon_{n+1})/2$ and $y_{n+1}' = (\varepsilon_0 - \varepsilon_{n+1})/2$. We can consider the conformal circle $\tilde{C}$ as the path (z_r) in M_n defined as $z_r = \overline{(\varepsilon r^2 x_0' + r\varepsilon_1 - y_{n+1}')}$.[150] Let us consider the following vectors:

$$\begin{cases} F_0(r) = A(\varepsilon r^2 x_0' + r\varepsilon_1 - y_{n+1}'), & \text{where } A \text{ denotes a nonzero function with values in } \mathbf{R}, \\ F_1(r) = \frac{d}{dr}(A^{-1}F_0) + B_1 F_0, & \\ F_{n+1}(r) = -A^{-1}x_0' - \frac{\varepsilon}{2}B_1 F_1 + \frac{\varepsilon}{4}B_1^2 F_0, & \text{where } B_1 \text{ denotes any function from } \mathbf{R} \text{ into } \mathbf{R}. \end{cases}$$

148 M. Berger, (a) *Géométrie*, vol. 1, Cedic Nathan, Paris, 1977, p. 121; or (b) *Géométrie différentielle*, A. Colin, Paris, 1972, pp. 79–82.

149 Elie Cartan, Les espaces à connection conforme, *Annals of the Polish Math. Soc.*, 2, 1923, pp. 171–221.

150 We recall that any element in the Möbius space is the class $\bar{z}$ of an element $z = \lambda x_0 + a + \mu y_{n+1}$ of E_{n+2} with $a \in E_n$ and $(\lambda, \mu) \in \mathbf{R}^n$ such that $\lambda\mu + a^2 = 0$.

The plane generated by F_0 and F_{n+1} is a hyperbolic plane and one can find $(n-1)$ vectors F_i, $i = 2, 3, \dots, n$, in E_{n+2} such that $\{F_1(r), F_2(r), \dots, F_n(r)\}$ is an orthonormal basis of $(\mathrm{Vect}\{F_0, F_{n+1}\})^{\perp}$ (the subspace of vectors orthogonal to $\mathrm{Vect}\{F_0, F_{n+1}\}$), the hyperbolic plane generated by F_0 and F_{n+1}. Then

$$z_r = \overline{F_0(r)}, \tag{g}$$

with the following fundamental relations:

$$\begin{aligned} \frac{dF_0}{dt} &= A_0 F_0 + A_1 F_1, \\ \frac{dF_1}{dt} &= A_2 F_0 - 2\varepsilon A_1 F_{n+1}, \\ \frac{dF_{n+1}}{dt} &= -A_0 F_{n+1} - \frac{\varepsilon}{2} A_2 F_1, \end{aligned} \tag{h}$$

where we put

$$\begin{aligned} A_0 &= \frac{dr}{dt}\left(\frac{dA}{dr}A^{-1} - AB_1\right), \\ A_1 &= A\frac{dr}{dt}, \\ A_2 &= \frac{dr}{dt}\left(\frac{dB_1}{dr} + B_1\frac{dA}{dt}A^{-1} - \frac{AB_1^2}{2}\right). \end{aligned} \tag{i}$$

First, Elie Cartan has given such relations in the given reference p. 206, in order to define what he has called "conformal circles in M_n."

Conversely, let us assume that there exists a path (z_u) in M_n and a basis $\{F_0(u), F_i(u), F_{n+1}(u)\}$ for $i = 1, 2, \dots, n$ in E_{n+2}, where $\{F_0, F_{n+1}\}$ defines a Witt basis of $\mathrm{Vect}\{F_0, F_{n+1}\}$ and $\{F_i(t)\}$ is an orthonormal basis of $(\mathrm{Vect}\{F_0, F_{n+1}\})^{\perp}$, the orthogonal of $\mathrm{Vect}\{F_0, F_{n+1}\}$, such that $\overline{F_0(u)} = z_u$ and equations of type (h) above are satisfied with $A_1 \neq 0$. Then we can find numerical functions A, B, and r that satisfy the relations (i) above.

Now, we have $(d^3/dr^3)(A^{-1}F_0) = 0$, that is $A^{-1}F_0 = (r^2/2)A' + rB' + C'$ where A', B', C' are fixed vectors in E_{n+2} such that $A'^2 = C'^2 = 0$, $B(A', B') = B(B', C') = 0$, where B is the fundamental bilinear symmetric form associated with the quadratic form defined on E_{n+2}. Moreover $B(A', C') + B'^2 = 0$ and $B'^2 \neq 0$ as $(d/dr)(A^{-1}F_0)$ is non isotropic. Thus

$$z_u = \overline{F_0(u)} = \left(\overline{\frac{r^2}{2}A' + rB' + C'}\right)$$

is the conformal circle

$$M_n \cap P(\mathrm{Vect}\{A', C'\} \oplus \mathrm{Vect}\, B').$$

Thus we find the previous definition.

2.11.1.6 Study of a Conformal Line

Let $P(F)$ be a projective subspace, where $F = \mathrm{Vect}\{a, b\}$ is a totally isotropic subspace of E_{n+2}. Put $y'_{n+1} = -b$. We can find a vector x'_0 in E_{n+2} such that $(x'_0)^2 = 0$, $B(x'_0, a) = 0$, $2B(x'_0, y'_{n+1}) = 1$ and construct an orthonormal basis $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ of $(\mathrm{Vect}\{x'_0, y'_{n+1}\})^{\perp}$, the subspace of vectors orthogonal to $\mathrm{Vect}\{x'_0, y'_{n+1}\}$ such that $(\varepsilon_1)^2 = \varepsilon$, $(\varepsilon_2)^2 = -\varepsilon$, with $\varepsilon = \pm 1$ and $a = \varepsilon_1 + \varepsilon_2$.

2.11.1.6.1 Equivalence of Such a Definition with That Given by E. Cartan in the Euclidean Case

First, we prove that the previous definition implies that of Elie Cartan. $P(F)$ can be viewed as the path (z_r) in M_n defined by $z_r = \overline{(ra - y'_{n+1})}$. Let us consider the following vectors:

$G_0(r) = A(ra - y'_{n+1})$, where A is nonzero numerical function defined on $\mathbf{R}$;
$G_1(r) = 2\varepsilon r x'_0 + \varepsilon_1 + B_1 G_0$, where B_1 is any numerical function;
$G_2(r) = 2\varepsilon r x'_0 - \varepsilon_2 + B_2 G_0$, where B_2 is any numerical function.

One can find $G_{n+1}(r) \in (\mathrm{Vect}\{G_1, G_2\})^{\perp}$ such that $2B(G_0, G_{n+1}) = 1$ and $(n-2)$ orthonormal vectors $G_i(r) \in (\mathrm{Vect}\{G_0, G_{n+1}, G_1, G_2\})^{\perp}$. Then we have

$$z_r = \overline{G_0(r)}, \tag{j}$$

and the following relations:

$$\begin{aligned} \frac{dG_0}{dt} &= D_0 G_0 + D_1(G_1 - G_2), \\ \frac{d}{dt}(G_1 - G_2) &= D_2 G_0 + D_3(G_1 - G_2), \text{ where } t \text{ is any parameter,} \end{aligned} \tag{k}$$

with

$$\begin{aligned} D_0 &= \frac{dr}{dt}\left(\frac{dA}{dr}A^{-1} - A(B_1 - B_2)\right), \\ D_1 &= A\frac{dr}{dt}, \\ D_2 &= \frac{d}{dt}(B_1 - B_2) + (B_1 - B_2)D_0, \\ D_3 &= \frac{dr}{dt}A(B_1 - B_2). \end{aligned} \tag{l}$$

First, Elie Cartan has given in the above reference (see 2.11.1.5.2), p. 204, the analytic definition—in the case that E_n is a Euclidean space—of what he has called "minimal lines" in M_n.

Conversely, let us assume that there exists a path (z_u) in M_n and a basis $\{G_0(u), G_i(u), G_{n+1}(u)\}$, $i = 1, 2, \dots, n$, in E_{n+2}, where the set $\{G_0, G_{n+1}\}$ defines a Witt basis of $\mathrm{Vect}\{G_0, G_{n+1}\}$ and $\{G_i(u)\}$, $i = 1, 2, \dots, n$, is an orthonormal basis of $(\mathrm{Vect}\{G_0, G_{n+1}\})^{\perp}$, such that $\overline{G_0}(u) = z_u$ and equations of type (k) above. One can find numerical functions A, B_1, B_2, and r satisfying the relations (l) above. Then, we have $(d^2/dr^2)(A^{-1}G_0) = 0$, that is, $A^{-1}G_0 = ra' + b'$ with $(a')^2 = (b')^2 = B(a', b') = 0$. $z_u = \overline{G_0(u)} = \overline{(ra' + b')}$ is then the "minimal line" $P(\text{Vect}\{a', b'\})$. Then we find the previous definition.

2.11.1.7 Definition Let $\tilde{P}$ be a principal fiber bundle on a n-dimensional manifold N, with structure group G and let $\tilde{w}$ be a conformal Cartan connection on $\tilde{P}$. Let $\bar{M}$ be the fiber bundle with typical fiber M_n associated with $\tilde{P}$. A path $\gamma : t \to \gamma(t)$ in N is called a "conformal circle," respectively a "minimal line," if its development in the fiber in N over a point of γ_t — the fiber that can be identified with M_n—is a "conformal circle," respectively a "minimal line."

Such a definition is intrinsic, as the image by an element of $PO(p+1, q+1)$ of a "conformal circle" is a "conformal circle," respectively a "minimal line."

2.11.1.8 Proposition *Oguie conformal geodesics are "conformal circles" or "minimal lines."*

Let us consider again the equations (e) and (f) above in 2.11.1.4.2. The equation (f) admits the unique solution $g_t = \exp((t - t_0)a)$, where exp is the classical exponential mapping defined on $po(p+1, q+1) = \mathcal{L}PO(p+1, q+1)$. Here, a is identified with an element of this Lie algebra. From results given above in 2.4.2.3, g_t appears as the equivalence class modulo $\pm\text{Id}_{E_{n+2}}$ of the element of $O(p+1, q+1)$ defined by

$$z \to (1 + (t - t_0)x_0 a)z(1 + (t - t_0)a x_0)$$

such that the path $y_t = g_t \bullet \bar{y}_{n+1}$ in M_n satisfies

$$y_t = g_t \bullet \bar{y}_{n+1} = \overline{(t - t_0)^2 a^2 x_0 + (t - t_0)a - y_{n+1}}.$$

We recognize the "conformal circle" $(M_n \cap P(\text{Vect}\{x_0, y_{n+1}\} \oplus \text{Vect}\{a\})$ if $a^2 \neq 0$ or the "minimal line" $P(\text{Vect}\{a, y_{n+1}\}$ if $a^2 = 0$, whence the result appears.

2.11.1.9 Proposition (i) *If a path (z_t) in N is a "conformal circle" or a "minimal line," it is a conformal geodesic up to a change of parameter.*

(ii) *Any path that can be deduced from a conformal geodesic by a change of parameter is a "conformal circle" or a "minimal line."*

Proof. (i) Let z_t be a "conformal circle" or a "minimal line" in N. Let $(\bar{p}_t)$ be the horizontal lift of (z_t) in $\bar{P}$ with origin $\bar{p}_0 \in \bar{P}_{\gamma_{t_0}}$. There exists a path (g_t) in $PO(p+1, q+1)$ such that $(\bar{p}_t \bullet g_t)$ is the image by i_1—defined in 2.11.1.4—of a path (φ_t) in $\tilde{P}$. The development of (z_t) in the fiber $\bar{M}_{\gamma_{t_0}}$ is then the path defined by $u_t = \bar{p}_0(g_t \bullet \bar{y}_{n+1})$ with $g_t^{-1} \bullet \dot{g}_t = \tilde{w}(\dot{\varphi}_t)$ (see 2.11.1.4.1 above, formulas (c_1) and (d)).

By assumption, the path defined by $y_t = g_t \bullet \bar{y}_{n+1}$ is a "conformal circle" or a "minimal line" in M_n. By using the remarks made after Definition 2.11.1.4.3, there exists a basis $\{\varepsilon_0, \varepsilon_{n+1}, \varepsilon_i\}$, $i = 1, 2, \ldots, n$, where $\{\varepsilon_0, \varepsilon_{n+1}\}$ is a Witt basis of

Vect$\{\varepsilon_0, \varepsilon_{n+1}\}$ and $\{\varepsilon_i\}$ is an orthonormal basis of $(\text{Vect}\{\varepsilon_0, \varepsilon_{n+1}\})^{\perp}$, such that

$$y(t) = \overline{r(t)^2 a'^2 \varepsilon_0 + r(t) a' - \varepsilon_{n+1}},$$

where a' is a vector in E_{n+2} and r a numerical function.

Let us consider $k_0 \in O(p+1, q+1)$ that sends the basis $\{x_0, e_i, y_{n+1}\}$ $(i = 1, 2, \ldots, n)$ onto the basis $\{\varepsilon_0, \varepsilon_i, \varepsilon_{n+1}\}$ $(i = 1, 2, \ldots, n)$. Then we have

$$y_t = \bar{k}_0 \bullet \overline{r(t)^2 a^2 x_0 + r(t) a - y_{n+1}},$$

where $a = k_0^{-1} \cdot (a') \in E_n$. Moreover, we have

$$\overline{r(t)^2 a^2 x_0 + r(t) a - y_{n+1}} = \exp(r(t) \cdot a) y_{n+1},$$

where exp is the exponential mapping defined on $\mathcal{L}PO(p+1, q+1) = po(p+1, q+1)$; $(r(t) \cdot a$ is identified with an element of that Lie algebra . Then we have

$$\begin{cases} y_t = \bar{k}_0 \bullet \exp(r(t) \cdot a) \bar{y}_{n+1}, \\ y_t = g_t \bullet \bar{y}_{n+1}. \end{cases}$$

Thus, there exists a path (h_t) in G_0 such that

$$\underline{g_t \bullet h_t = \bar{k}_0 \bullet \exp(r(t) a)}. \tag{m}$$

Let (ψ_t) be the path in $\tilde{P}$ defined by $\psi_t = \varphi_t \bullet h_t$. It is a lift of the path (z_t) in $\tilde{P}$ that satisfies

$$\begin{aligned} \underline{\tilde{w}(\dot{\psi}_t)} &= \tilde{w}(\dot{\varphi}_t \bullet h_t + \varphi_t \bullet \dot{h}_t) = adh_t^{-1} \bullet \tilde{w}(\dot{\varphi}_t) + h_t^{-1} \bullet \dot{h}_t \\ &= adh_t^{-1}(g_t^{-1} \bullet \dot{g}_t) + h_t^{-1} \bullet \dot{h}_t = \underline{r'(t) a}, \end{aligned}$$

by (m). It is the integral curve of a basic field on $\tilde{P}$, up to a change of parameter, whence the result.

(ii) Let $z : t \to z_t$ be an Oguie conformal geodesic. There exists a lift $\varphi : t \to \varphi_t$ of z in $\tilde{P}$ such that $\tilde{w}(\dot{\varphi}_t) = a$, a being a fixed vector in $\mathbf{R}^n$. Let λ be a numerical function and let $\tilde{\gamma} = \gamma \circ \lambda$ and $\tilde{\varphi} = \varphi \circ \lambda$. Let $(\bar{p}_t)$ be the horizontal lift of $\tilde{z}$ in $\tilde{P}$ with origin $i_1(\tilde{\varphi}_{t_0})$; there exists a path g_t in $PO(p+1, q+1)$ such that $\bar{p}_t \bullet g_t = i_1(\bar{\varphi}_t)$. By using formula ($c_1$) in 2.11.1.4.1 above, the development of $\tilde{z}$ in the fiber $M_{\tilde{z}_{t_0}}$ is the path (u_t) defined by $u_t = i_1(\tilde{\varphi}_{t_0})(g_t \bullet \bar{y}_{n+1})$, where the path (g_t) satisfies $g_t^{-1} \bullet \dot{g}_t = \tilde{w}(\dot{\tilde{\varphi}}_t) = \lambda'(t) a$. Since $g_{t_0} = e$ (the unit element in $PO(p+1, q+1)$), we have

$$g_t = \exp((\lambda_t - \lambda_{t_0}) a),$$

where exp is the classical exponential mapping defined on

$$\mathcal{L}PO(p+1, q+1) = po(p+1, q+1).$$

The path $y_t = g_t \bullet \bar{y}_{n+1}$ satisfies the relation

$$y_t = \overline{(\lambda_t - \lambda_{t_0})^2 a^2 x_0 + (\lambda_t - \lambda_{t_0}) a - y_{n+1}},$$

that is, it is a "conformal circle" or a "minimal line," whence the result.

2.11.2 Generalized Conformal Connections

We want to present the construction made by J. L. Milhorat in his thesis.[151] Milhorat finds again the conformal connections found by R. Hermann[152] by an intrinsic method using Greub extensions of structures. As already seen in Section 2.4, E_n can be identified with an open set of the Möbius space M_n, and the action of an element of the Möbius group $PO(p+1, q+1)$ on M_n induces a conformal transformation of E_n.

Let $\xi = (M, \pi, N)$ be a pseudo-riemannian bundle with typical fiber E_n; J. L. Milhorat constructs a bundle ζ with typical fiber M_n such that ξ can be identified with a subbundle of ζ. A conformal Ehresmann connection on the bundle ζ defines a horizontal subbundle of the bundle ξ, called a "generalized conformal connection." Such generalized conformal connections are effectively the Hermann conformal connections.

2.11.2.1 Preliminary Definitions

2.11.2.1.1 Definitions

- Let $\xi = (M, \pi, N)$ be a bundle with typical fiber E_n over a manifold N of dimension n. We assume that this bundle is pseudo-riemannian, i.e., that there exists a vector field $g : x \to g_x$ that assigns a nondegenerate symmetric bilinear form of type (p, q), $g_x : \pi^{-1}(x) \times \pi^{-1}(x) \to \mathbf{R}$ to any x in N. $(p+q = n, n > 2)$. Therefore, there exists a trivializing atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ of ξ such that $\varphi_{\alpha,x} : E_n \to M_x$ belong to the classical pseudoorthogonal group $O(p, q)$ that is the Lie structure group of ξ. The corresponding transition functions are $g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(p, q)$.
- Let $\xi_1 = (M_1, \pi_1, N)$ be the bundle Whitney sum of the bundle ξ and the trivial bundle $\xi_2 = (N \times E_2, \pi_2, N)$, where E_2 is the standard hyperbolic plane $E_{1,1}$ provided with a bilinear symmetric form g_2 of type $(1, 1)$. ξ_1 is a bundle with typical fiber E_{n+2} and a pseudo-riemannian bundle corresponding to the fundamental bilinear symmetric form of type $(p+1, q+1)$ g_1 defined by

$$g_1(x; z, w) = \begin{cases} g_x(z, w), & \text{for any } z, w \in M_x, \\ 0, & \text{for any } z \in M_x,\ w \in \{x\} \times E_2, \\ 0, & \text{for any } z \in \{x\} \times E_2,\ w \in M_x, \\ g_2(z, w), & \text{for any } z, w \in \{x\} \times E_2 \simeq E_2. \end{cases}$$

  The mappings $\psi_{\alpha,x} : E_{n+2} \to (M_1)_x$, $\psi_{\alpha,x} \in O(p+1, q+1)$ such that $\psi_{\alpha,x} = \varphi_{\alpha,x}$ on E_n and $\psi_{\alpha,x} = \mathrm{Id}_{E_2}$ on E_2 define a trivializing atlas $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ of ξ_1 with transition functions $j \circ g_{\alpha\beta}$, where j denotes the classical injective mapping from $O(p, q)$ into $O(p+1, q+1)$.

[151] J. L. Milhorat, Sur les connections conformes, Thesis, Université Paul Sabatier, Toulouse, 1985.

[152] R. Hermann, *Gauge Fields and Cartan–Ehresmann Connections, Part A*, Math. Sci. Press, Brookline, 1945.

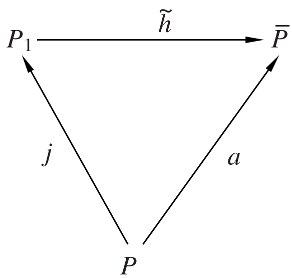
- Let P be the principal fiber bundle of the frames associated with ξ, the $(g_{\alpha\beta})$ are the cocycles of P for the trivializing atlas $(U_\alpha)_{\alpha\in A}$—and let P_1 be the j-extension of P, i.e., the bundle with typical fiber $O(p+1, q+1)$ associated with P, where $O(p, q)$ acts on the left on $O(p+1, q+1)$ via the morphism of Lie groups j. P_1, principal bundle with cocycles the $(j \circ g_{\alpha\beta})$, is by definition the principal bundle of frames associated with ξ_1. We agree to denote by the same letter j the morphism from P into P_1 defined by

$$j(P) = \overline{(p, \mathrm{Id}_{E_{n+2}})}, \qquad \text{for any } p \in P.$$

Following the diagram given above in 2.5.1.2, we denote by $\tilde{h}$ the canonical mapping from $O(p+1, q+1)$ onto $PO(p+1, q+1)$ and denote by $\bar{P}$ the $\tilde{h}$-extension of the bundle P_1. $\bar{P}$ is the principal bundle with cocycles the $(\tilde{h} \circ j \circ g_{\alpha\beta})$ for the trivializing atlas $(U_\alpha)_{\alpha\in A}$. We agree to denote by $\tilde{h}$ the morphism from P_1 into $\bar{P}$ defined by

$$\tilde{h}(p_1) = \overline{(p_1, e)},$$

where e denotes the identity element in $PO(p+1, q+1)$.
- Let us consider $a = \tilde{h} \circ j$, which is a canonical injective mapping from $O(p, q)$ into $PO(p+1, q+1)$. The a-extension of P has for cocycles the $a \circ g_{\alpha\beta} = \tilde{h} \circ j \circ g_{\alpha\beta}$ and is isomorphic to the principal bundle $\bar{P}$. If we agree to denote by a the principal morphism from P into $\bar{P}$ defined by $a(p) = \overline{(p, e)}$, we have the following commutative diagram:



- Let $\bar{M}$ denote the set $\bar{M} = \cup_{x\in N} \bar{M}_x$, where the $\bar{M}_x$ are the Möbius spaces associated with the fibers $(M_1)_x$ of ξ_1.
- Let $\bar{\pi}$ be the mapping from $\bar{M}$ onto N defined by $\bar{\pi}(\bar{z}_x) = x$, for any $\bar{z}_x \in M_x$. Then we have the following result.

2.11.2.1.2 Theorem *$\bar{M}$ is endowed with the structure of a differentiable bundle with projection $\bar{\pi}$ and typical fiber M_n, the Möbius n-dimensional space.*

The demonstration will be given below in the exercises.

2.11.2.2 Generalized Conformal Connections on a Pseudo-Riemannian Bundle

We use the terminology of Greub, Halperin, and Vanstone[153] (chapter VII, §6). Let α be the isomorphism of vectorial bundles $\alpha : VM \to M \times_N M$, where $M \times_N M$ is the fiber product of M with itself and VM, defined in 2.8.7 is the vertical subbundle of TM (α identifies the space tangent to the fiber M_x at $z \in M_x$ with M_x) given in the exercise associated with section 2.8.8.8 and $\bar{\varphi}$ is the morphism of differential bundles given in the exercise associated with 2.11.2.1.2.[154]

2.11.2.2.1 Definition Let $\eta = (M, \pi, N)$ be a vectorial bundle and let $H(M)$ be a horizontal subbundle of $T(M)$ and let $K_M : T(M) \to V(M)$ be the corresponding projection. The mapping D from the module $\Gamma(\eta)$ of sections of η into the module of 1-forms on N with values in η, defined by $D_S = \alpha \circ K_M \circ ds$, $s \in \Gamma(\eta)$, is called by definition a generalized connection on η associated with $H(M)$.

Such a definition is a generalization of the definition of a linear connection of the vector bundle η. Moreover, we do not assume any hypothesis of Ehresmann type on the horizontal subbundle $H(M)$.

2.11.2.2.2 Definition Let $\xi = (M, \pi, N)$ be a pseudo-riemaniann vectorial bundle and let $\zeta = (\bar{M}, \bar{\pi}, N)$ be the bundle, the fibers of which $\bar{M}_x$, $x \in N$, are the Möbius spaces associated with the $(M_x \oplus E_2)$. Since the morphism $\bar{\varphi}$ defined in 2.11.2.1.2 from ξ into ζ is a local diffeomorphism, any horizontal subbundle $H(\bar{M})$ of $T(\bar{M})$ "induces" a horizontal subbundle $H(M)$ of $T(M)$ and therefore a generalized connection on ξ. In particular, if $H(\bar{M})$ is a conformal Ehresmann connection, then by definition, the generalized connection associated with $H(M)$ is called conformal.

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[153] Greub, Halperin, Vanstone, *Connections, Curvature and Cohomology*, vol. 2, Academic Press, 1972.

[154] • α is the isomorphism of vector bundles $\alpha : V(M) \to M$ induced by the identification between the fiber M_x at a point x of N and the tangent space $T_z(M_x)$ to M_x at a point z of M_x; we can define the covariant derivative ∇ associated with $H(M)$ (cf. Definition 2.8.4) as a mapping from the module $\Gamma(M)$ of sections of the vector bundle (M, π, N, F) into the module of 1-forms on N with values in the vector bundle M, by putting $\nabla s = \alpha \circ K_M \circ ds$, $s \in \Gamma(M)$, where K_M denotes the projection $T(M) \to V(M)$ associated with $H(M)$.

• $\bar{\varphi}$ is the following morphism of differentiable fibrations from ξ into ζ. Let $(U, \hat{\varphi})$ be the local chart introduced in 2.9.1.3.1 above. For any $y \in E_n$, we put $\hat{\varphi}^{-1}(y) = \overline{u(y)}$, where u is defined by $u(y) = y^2 x_0 + y - y_{n+1}$. Let $y_x \in M_x \bullet p \in P_x$ and $y \in E_n$ such that $p(y) = y_x$. We put $\bar{\varphi}(y_x) = i(p)(\overline{u(y)})$; $i(p)$ can be considered an element of the principal bundle associated with ζ, that is, as a diffeomorphism from M_n onto $\bar{M}_x$.

Such a definition is intrinsic, since if $p' \in P_x$ and $y' \in E_n$ are such that $p'(y') = p(y)$, then there exists $g \in O(p, q)$ such that $p' = p \bullet g$ and $g' = g^{-1}(y)$. According to 2.4.2.2, 2.9.1.3.1 and 2.9.1.3.3 above, we have $u(y') = u(g^{-1} \bullet y) = j(g^{-1}) \bullet u(y)$ (cf. 2.11.2.1.1), whence $i(p')\overline{u(y')} = i(p) \bullet i(g)[\tilde{h} \circ j(g^{-1}) \bullet \overline{u(y)}] = i(p) \bullet \overline{u(y)}$, as $i = \tilde{h} \circ j$, with previous notation. Thus, the mapping $\bar{\varphi} : M \to \bar{M}$ that sends fiber into fiber is an N-morphism of differentiable fibrations from ξ into ζ. Since $\bar{\varphi}$ is locally the mapping $\hat{\varphi}^{-1} : E_n \to M_n$, the conclusion follows.

We recall that u from $E_n(p,q)$ into the isotropic cone of $E_{n+2}(p+1,q+1)$ is such that $u(z) = z^2 x_0 + z - y_{n+1}$.

Let us define u_M as the mapping from M into M_1, by (1) (cf. 2.11.2.1.1 Definitions),

$$u_M(y_x) = j(p) \circ u \circ p^{-1}(y_x),\ y \in M_x,\ p \in P_x. \tag{1}$$

Such a definition is intrinsic since if p and p' belong to the fiber at $x \in N$ of the principal bundle P of frames, there exists g in $O(p,q)$ such that $p' = p.g$ (cf. 2.11.2.2 and the corresponding exercise) and

$$u \circ p'^{-1}(y_x) = j(g^{-1}) \circ u \circ p^{-1}(y_x),$$

whence we find that

$$j(p') \circ u \circ p'^{-1}(y_x) = j(p).j(g)[j(g^{-1}) \circ u \circ p^{-1}(y_x)] = j(p)u \circ p^{-1}(y_x).$$

The mapping $u_M : M \to M_1$, which maps a fiber into a fiber, is therefore by definition an N-morphism of differential bundles from ξ into ξ_1. We have the following result.

2.11.2.2.3 Proposition *Let $\xi = (M, \pi, N)$ be a vector pseudo-riemannian bundle. We can bijectively associate with any generalized conformal connection D corresponding to a horizontal subbundle $H(M)$ of $T(M)$ a linear pseudo-riemannian connection ∇ on the bundle $\xi_1 = \xi \oplus \xi_2 = (M_1, \pi_1, N)$ associated with a horizontal subbundle $H(M_1)$ of $T(M_1)$ such that*

- *if (z_t) is a horizontal path of M, relative to $H(M)$, there exists a path (λ_t) of $\mathbf{R}^*$ such that the path $(\lambda_t u_M(z_t)$ of M_1 is horizontal relative to $H(M_1)$.*
- *Conversely, if (z_t) is a path of M that satisfies the condition there exists a path (λ_t) of $\mathbf{R}^*$ such that the path $(\lambda_t u_M(z_t))$ of M_1 is horizontal relative to $H(M_1)$, then the path (z_t) is horizontal, relative to $H(M)$.*

Proof. Let $H(M)$ be a horizontal subbundle of $T(M)$ that defines a conformal generalized connection. By definition, $H(M)$ is induced by a conformal Ehresmann connection $H(\bar{M})$ on the bundle $\zeta = (\bar{M}, \bar{\pi}, N)$, the fibers of which are the Möbius spaces associated with the fibers of ξ_1.

According to 2.8.8.4, the connection $H(\bar{M})$ is associated with a principal connection with form $\bar{w}$ on the principal fiber bundle associated with ζ, isomorphic, as already seen, to $\bar{P}$, the $\tilde{h}$-extension of the principal fiber bundle of frames associated with ξ_1. We need now the following lemma:

2.11.2.2.4 Lemma *We can bijectively associate with any principal connection with form $\bar{w}$ on $\bar{P}$ a principal connection with form σ on the principal bundle P_1 such that*

$$\sigma = (dh)_{e'}^{-1}.\tilde{h}^*(\bar{w}),$$

where

$$e' = \mathrm{Id}_{E_{n+2}}. \tag{2}$$

Proof.

- If $\bar{w}$ is the form corresponding to a principal connection on $\bar{P}$, then the 1-form σ defined by (2) is the form corresponding to a principal connection on P_1.
- Conversely, if σ is the form of a principal connection on P_1, there exists a principal connection with form $\bar{w}$ on $\bar{P}$ such that $\bar{w}$ satisfies the relation (2).[155]
- We can now come back to the proof of Proposition 2.11.2.2.3. The principal connection with form $\bar{w}$, which defines the conformal Ehresmann connection $H(\bar{M})$ on ξ, is associated, according to Lemma 2.11.2.2.4, with a unique principal connection $H(P_1)$ on P_1. According to a classical result,[156] $H(P_1)$ defines on ξ_1 a linear pseudo-riemannian connection ∇.
- Conversely, any linear pseudo-riemannian connection ∇ on ξ_1 is induced by a unique principal connection $H(P_1)$ on P_1. Let $H(\bar{P})$ be the principal connection bijectively associated with $H(P_1)$ according to Lemma 2.11.2.2.4. $H(\bar{P})$ induces a conformal Ehresmann connection on ξ and then a generalized conformal connection D on ξ.

2.11.2.2.5 Characterization

- Let (y_t) be a horizontal path of M relative to $H(M)$. By definition, the path $(\bar{\varphi}(y_t))$ is a horizontal path of $\bar{M}$ for the conformal Ehresmann connection, where $\bar{\varphi}$ is the morphism of differential bundles defined by

  $$\bar{\varphi}(y_x) = i(p)(\overline{u(y)}),\ \ y_x \in M_x,\ p \in P_x,\ \text{ and } y \in E_n \text{ such that } p(y_x) = y_x$$

  (cf. the exercise associated with 2.11.2.1.2).
- Let y be a fixed element in E_n. There exists a path (φ_t) in P such that $y_t = \varphi_t(y)$. We have

  $$\bar{\varphi}(y_t) = i(\varphi_t)(\overline{u(y)}). \tag{3}$$

- Let $(\gamma_t = \pi(y_t))$ be the projection of the path (y_t) onto N and let (ψ_t) be the horizontal lift of the path (γ_t) in P_1 with origin (ψ_{t_0}), a point in the fiber of P_1 over γ_{t_0} (for the connection with form σ on P_1). According to (2), $(h(\psi_t))$ is the horizontal lift of (γ_t) in $\bar{P}$ with origin $\tilde{h}(\psi_{t_0})$ for the connection with form $\bar{w}$ on $\bar{P}$.

  Since the path $(\bar{\varphi}(y_t))$ is a horizontal path in $\bar{M}$, we have

  $$\bar{\varphi}(y_t) = \bar{\tau}_\gamma.\bar{\varphi}(y_{t_0}),$$

where $\bar{\tau}_\gamma$ is the parallel displacement in $\bar{M}$ associated with the path (γ_t) corresponding to the conformal Ehresmann connection $H(\bar{M})$. According to the previous

[155] Kobayashi S. and Nomizu K., *Foundations of Differential Geometry*, vol. 1, Interscience Publishers, New York, 1963, Proposition 6.1, p. 79.

[156] Pham Mau Quan, *Introduction à la Géométrie des Variétés Différentiables*, Dunod, Paris, 1969, Théorème 1, p. 267.

Remark 2.8.4.2.7, we have

$$\bar{\varphi}(y_t) = \tilde{h}(\psi_t) \circ \tilde{h}(\psi_{t_0})^{-1} \bullet \bar{\varphi}(y_{t_0}),$$

i.e., according to (3),

$$i(\varphi_t)(\overline{u(y)}) = \tilde{h}(\psi_t) \circ \tilde{h}(\psi_{t_0})^{-1}.i(\varphi_{t_0})(\overline{u(y)}). \tag{4}$$

Therefore, y being a fixed element of E_n and (φ_t) being the path in P such that $y_t = \varphi_t(y)$, *the proposition "(y_t) is a horizontal path of M, relative to $H(M)$" is equivalent to* (4). Moreover, there exists a path (g_t) in $O(p+1, q+1)$ such that

$$j((\varphi_t) = \psi_t \cdot g_t. \tag{5}$$

Since $\tilde{h} \circ j = i$, (4) is equivalent to

$$\tilde{h}(g_t) \bullet \overline{u(y)} = \tilde{h}(g_{t_0}) \bullet \overline{u(y)},$$

which is also equivalent to the following proposition: "*There exists a path (λ_t) of* $\mathbf{R}^*$ *such that*

$$\lambda_t g_t \bullet u(y) = g_{t_0} \bullet u(y)." \tag{6}$$

Moreover, $(\lambda_t g_t \bullet u(y) = g_{t_0} \bullet u(y))$ is equivalent, according to (5), to

$$\lambda_t j(\varphi_t) \bullet u(y) = \psi_t \circ \psi_{t_0}^{-1}[j(\varphi_{t_0}) \bullet u(y)],$$

i.e., according to the definition of the morphism u_M of differential fiber bundles, (1),

$$\lambda_t u_M(y_t) = \psi_t \circ \psi_{t_0}^{-1}(\lambda_{t_0}^{-1} u_M(y_t)). \tag{7}$$

According to Remark 2.8.4.2.7, (7) is equivalent to the proposition, "*the path $(\lambda_t u_M(y_t))$ of M_1 is horizontal, relative to $H(M_1)$*."

Thus, (6) is equivalent to the proposition, "*There exists a path (λ_t) of* $\mathbf{R}^*$ *such that the path $(\lambda_t u_M(y_t))$ of M_1 is horizontal*."

We have obtained the following characterization:

The following propositions are equivalent:

(i) *The path (γ_t) in M is horizontal, relative to $H(M)$.*
(ii) *There exists a path (λ_t) in* $\mathbf{R}^*$ *such that the path $(\lambda_t u_M(y_t))$ of M_1 is horizontal, relative to $H(M_1)$.*

2.11.2.2.6 Corollary *Let D be a generalized conformal connection on the pseudo-riemannian fiber bundle $\xi = (M, \pi, N)$ and let ∇ be the linear pseudo-riemannian connection on the associated fiber bundle $\xi_1 = \xi \oplus \xi_2$ (Proposition 2.11.2.2.3). The two following propositions are equivalent:*

(i) *For any s in $\Gamma(\xi)$, $D_s = 0$.*

(ii) *There exists a nonzero C^∞ function λ_s on N such that*

$$\nabla(\lambda_s \cdot u_M \circ s) = 0. \tag{8}$$

Let $s \in \Gamma(\xi)$, $X \in T(N)$. $D_X(s) = 0$ is equivalent to the condition, "*The path* $s(x_t)$ *is horizontal,* $\gamma : t \to x_t$ *being an integral curve of* X," which is equivalent to, "*there exists a path* (λ_t) *of* $\mathbf{R}^*$ *such that the path* $(\lambda_t u_M \circ s(x_t))$ *is horizontal*" according to Proposition 2.11.2.2.3, which is equivalent to, "*There exists a nonzero* C^∞ *function on* N *such that* $\nabla_X \lambda_s \cdot u_M \circ s = 0$." (For the necessary condition we assume that γ is defined on a compact $\mathcal{I}$ of $\mathbf{R}$ and that γ is simple and regular.) Therefore, there exists a function λ_s on N such that $\lambda_s(x_t) = \lambda_t$, for any $t \in \mathcal{I}$.

2.11.2.2.7 Local Expression of a Generalized Conformal Connection D on a Pseudo-Riemannian Fiber Bundle

Let (f_i), $i = 1, 2, \ldots, n$, be a moving frame, not necessarily orthonormal in the bundle ξ. The basis of local sections $\{x_0, f_i, y_{n+1}\}$, $i = 1, 2, \ldots, n$, constitutes a moving frame of the bundle $\xi_1 = \xi \oplus \xi_2$.

Let (σ^α_β), $\alpha, \beta = 0, 1, 2, \ldots, n, n+1$, be the components in the moving frame $\{x_0, f_i, y_{n+1}\}$ of the connection form associated with the pseudo-riemannian connection ∇ on ξ_1. Since $\nabla_{g_1} = 0$, these components satisfy the conditions

$$\begin{cases} \left.\begin{array}{l} \sigma^0_{n+1} = \sigma^{n+1}_0 = 0 \quad \sigma^0_i = -2 g_{ik}\sigma^k_{n+1} \\ \sigma^0_0 + \sigma^{n+1}_{n+1} = 0 \quad\ \ \sigma^{n+1}_i = -2 g_{ik}\sigma^k_0 \end{array}\right\} \qquad i, k = 1, 2, \ldots, n, \\ \sigma^i_j g_{ik} + g_{ji}\sigma^i_k = dg_{jk};\ i, j, k = 1, 2, \ldots, n \ (g_{ij} = g(f_i, f_j)), \end{cases} \tag{9}$$

with obvious notation. Let s be a local section of ξ defined by $s = s^i f_i$ and such that $D_s = 0$. Since $u_M \circ s = s^2 x_0 + s - y_{n+1}$, with $s^2 = g(s, s)$, according to (8), there exists a nonzero C^∞ function λ_s on N such that

$$\nabla \lambda_s (s^2 x_0 + s - y_{n+1}) = 0.$$

Explicitly, we obtain

$$\begin{cases} d\lambda_s s^2 + \lambda_s(ds^2 + s^2\sigma^0_0 + \sigma^0_j s^j) = 0, & \text{(a)} \\ d\lambda_s s^i + \lambda_s(s^2\sigma^j_0 - \sigma^i_{n+1} + ds^i + \sigma^i_j \sigma^j) = 0, & \text{(b)} \\ d\lambda_s + \lambda_s(\sigma^0_0 + \sigma^{n+1}_j s^j) = 0. & \text{(c)} \end{cases}$$

Equation (c) gives $\lambda_s^{-1} d\lambda_s$. From (a) and (c), we can deduce, taking account of relations (9),

$$ds^i - \sigma^i_{n+1} + \sigma^i_j s^j + \sigma^0_0 s^i - \sigma^0_{n+1}\left(\frac{1}{2} g^{ji} s^2 - s^j s^i\right) = 0. \tag{10}$$

Thus we have obtained the expression of $(D_s)^i$ in the left part of the equality (10). Equation (a) is always verified if we have (b) and (c), since (b) and (c) give

$$ds^2 + 2s^2\sigma^0_0 + \sigma^0_j s^j + s^2\sigma^{n+1}_j s^j = 0, \tag{11}$$

equality is always satisfied: from $\nabla_{g_1} = 0$, we can deduce that

$$0 = \nabla_{g_1}(s, s) = d[g(s, s)] - 2g_1(\nabla_s, s) = ds^2 - 2g_1(\nabla_s, s).$$

If we develop the last equality, taking account of (10), we obtain (11). Thus we have obtained the following result:

2.11.2.2.7.1 Proposition *Let D be a generalized conformal connection on a pseudo-riemannian fiber bundle $\xi = (M, \pi, N)$. Let $\{f_i\}$, $i = 1, 2, \ldots, n$, be a moving frame of the bundle ξ. We can write*

$$D_s = \left(ds^i + w^i + w^i_j s^j + w_0 s^i + w^0_j \left(\frac{1}{2} g^{ji} s^2 - s^j s^i \right) \right) f_i, \quad s = s^i f_i, \quad (12)$$

where the (w^i, w^i_j, w_0, w^0_j) are $(n+1)(n+2)/2$ local 1-forms on N such that

$$w^i_j g_{ik} + g_{ji} w^i_k = dg_{jk}, \quad \text{where } g_{jk} = g(f_j, f_k). \quad (13)$$

These local 1-forms are called the local 1-forms in the moving frame $\{f_i\}$ of the generalized connection D.

2.11.2.2.8 Proposition *Let D be a generalized conformal connection on a pseudo-riemannian fiber bundle $\xi = (M, \pi, N)$. Let $\{f_i\}$ and $\{f'_i\}$ be moving frames of ξ such that on the intersection U of their domains of definition we have $f'_i = A^j_i f_j$, where the (A^j_i) are C^∞ functions on U, the matrix with coefficients $(A^i_j(x))$ being an element of $GL(n)$ for any x in U. Let $\{w^i, w^i_j, w_0, w^0_j\}$ and $\{w'^i, w'^i_j, w'_0, w'^0_j\}$ be respectively the local 1-forms of the generalized conformal connection in the moving frames $\{f_i\}$ and $\{f'_i\}$ respectively. We have*

$$w'^i = (A^{-1})^i_j w^j, \; w'^i_j = (A^{-1})^i_k w^k_i A^i_j + (A^{-1})^i_j dA^k_j, \; w'_0 = w_0, \; w'^0_j = w^0_j A^i_j. \quad (14)$$

Conversely, if we consider a covering (U_α) of N, where (U_α) is the domain of definition of a moving frame of ξ and on each U_α if we consider 1-forms $\{w^i, w^i_j, w_0, w^0_j\}$ that satisfy the relations (13) and relations (14) on the intersections $U_\alpha \cap U_\beta$, we define a generalized conformal connection on ξ.

The proof will be given in the exercises.

2.11.2.2.9 Fundamental Remarks

According to Propositions 2.11.2.2.7 and 2.11.2.2.8, we can define a generalized conformal connection on a pseudo-riemannian fiber bundle by considering a mapping D locally defined by the relations (12) with local 1-forms $\{w^i, w^i_j, w_0, w^0_j\}$ that satisfy the relations (13) and transformation formulas such as (14) by changing the moving frame. But such generalized conformal connections are not generally Ehresmann connections. In order that a generalized conformal connection be an Ehresmann connection, we need to find, for any path (x_t) in N defined on an interval I of $\mathbf{R}$, a horizontal lift (y_t) of (x_t) in M, defined on I.

If we consider local coordinates $\{y^i\}$, $i = 1, 2, \ldots, n$, on M associated with a moving orthonormal frame $\{e_i\}$ of ξ, the horizontal lift (y_t) of (x_t) needs to satisfy the equations

$$\frac{dy^i}{dt} = -w^j(x'_j) - w^i_j(x'_j) y^j - w_0(x'_j) y^i - w^0_j(x'_j) \left(\frac{1}{2} g^{ji} y^2 y^j y^i \right) \quad (15)$$

following the relations (12) above.

Locally, we are led to the search of the integral curve, defined on all I, of an infinitesimal conformal transformation on E_n (See 2.9.1.3.3.). Now, some of those infinitesimal conformal transformations, for example those that correspond to conformal special transformations (or transversions), generate only a local parameter group. We cannot generally prove that there exists a solution of (15) defined on all I.

2.11.2.3 Curvature of a Generalized Conformal Connection

2.11.2.3.1 Definition Let D be a generalized conformal connection on a pseudo-riemannian fiber bundle $\xi = (M, \pi, N)$. Let $H(M)$ be the horizontal subbundle of $T(M)$ associated with D. According to Definition 2.8.7.6, the curvature associated with the subbundle $H(M)$ is the mapping (15),

$$\Omega : T(N) \times T(N) \to V(M) : (X, Y) \to \Omega(X, Y) = \Gamma([X, Y]) - [\Gamma(X), \Gamma(Y)],$$

where Γ is the horizontal morphism associated with $H(M)$.

2.11.2.3.2 Characterization

Let us consider the local coordinates $\{y^i\}, i = 1, 2, \ldots, n$, of M associated with a moving frame $\{f_i\}$ of ξ. According to the results of Section 2.8.7 above, we can give the local expression of Ω in such a moving frame by using the 2-forms $d\theta^i$, where the

$$\theta^i = dy^i + w^i + w^i_j y^j + w_0 y^i + w^0_j \left(\frac{1}{2} y^2 g^{ji} - y^j y^i\right)$$

constitute a local basis of $P(H(M))$. We apply an integrality condition.
We find that

$$\Omega = \left\{\Omega^i + \Omega^i_j y^j + \Omega_0 y^i + \Omega^0_j \left(\frac{1}{2} g^{ji} y^2 - y^j y^i\right)\right\} \frac{\partial}{\partial y^i}, \tag{16}$$

with

$$\Omega^i = dw^i + (w^i_j + w_0 \delta^i_j) \wedge w^j, \tag{17}$$

$$\Omega^i_j = dw^i_j + w^i_k \wedge w^k_j + w^i \wedge w^0_j + w^i_0 \wedge w_j, \tag{18}$$

$$\Omega_0 = dw_0 - w^0_k \wedge w^k, \tag{19}$$

$$\Omega^0_j = dw^0_j + w^0_k \wedge (w^k_j + w_0 \delta^k_j). \tag{20}$$

2.11.2.3.3 Remark Let ∇ be the pseudo-riemannian connection on $\xi_1 = \xi \oplus \xi_2$ bijectively associated with D according to Proposition 2.11.2.2.3. According to (10), the local 1-forms $\{w^i, w^i_j, w_0, w^0_j\}$ of the generalized connection D in the moving frame $\{f_i\}$ of ξ are the components $\{-\sigma^i_{n+1}, \sigma^i_j, \sigma^0_{n+1}, -\sigma^{n+1}_j\}$ of the form of connection associated with ∇ in the moving frame $\{x_0, f_i, y_{n+1}\}$ of ξ_1. It results from equations (17), (18), (19), (20) that the $\{\Omega^i, \Omega^i_j, \Omega_0, \Omega^0_j\}$ are the components $\{-\Sigma^i_{n+1}, \Sigma^i_j, \Sigma^0_{n+1}, -\Sigma^{n+1}_j\}$ of the curvature form of ∇ in the moving frame $\{x_0, f_i, y_{n+1}\}$ of ξ_1. For example,

$$\Omega^i = dw^i + (w^i_j + w_0\delta^i_j) \wedge w^j = -(d\sigma^i_{n+1} + \sigma^i_k \wedge \sigma^k_{n+1} + \sigma^i_{n+1} \wedge \sigma^{n+1}_{n+1})$$
$$= -\Sigma^i_{n+1}.$$

2.11.2.3.4 Study of the Peculiar Case of a Pseudo-Riemannian Manifold N with a Scalar Product of Signature (p, q)

This study will be given in the exercises below.

2.11.2.3.5 Applications

Two examples will be given in the exercises. The first one corresponds to the study of a conservative dynamical system with holonomic complete constraints with n degrees of freedom, satisfying the hypothesis of Painlevé. The second one concerns the equations of a charged particle in an electromagnetic field in classical general relativity.

2.12 Vahlen Matrices[157]

2.12.1 Historical Background[158]

In 1902,[159] K. Theodor Vahlen initiated the study of Möbius transformations of vectors in $\mathbf{R}^n$ by 2×2 matrices with entries in the Clifford algebra $C_{0,n}$. Such a study was reinitiated by L.V. Ahlfors.[160] A more precise study has been given by J. G. Maks.[161] Such matrices are used by J. Ryan[162] in Clifford analysis.

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[157] See, for example, chapter 19 of the excellent book by the late Pertti Lounesto, *Clifford Algebras and Spinors* second edition, Cambridge University Press, London Mathematical Society, Lecture Notes Series, 286, 2001.

[158] Cf. Appendix: A history of Clifford algebras in the previous book of P. Lounesto.

[159] Vahlen K. Th., Über Bewegungen und complexen Zahlen, *Math. Ann.*, 55, pp. 585–593, 1902.

[160] L. V. Ahlfors, (a) Old and new in Möbius groups, *Ann. Acad. Sci. Fenn., serie A.1 Math.*, 9, pp. 93–105, 1984. (b) Möbius transformations and Clifford numbers, pp. 65–73 in I. Chavel and M. M. Farkas (eds.), *Differential Geometry and Complex Analysis*, Springer, Berlin, 1985. (c) Möbius transformations in $\mathbf{R}^n$ expressed through 2×2 matrices of Clifford numbers, *Complex Variables Theory Appl.*, 5, pp. 215–224.

[161] J. G. Maks, Modulo (1, 1) periodicity of Clifford algebras and the generalized (anti-) Möbius transformations, Thesis, Technische Universiteit, Delft., 1989.

[162] J. Ryan, (a) Conformal Clifford manifolds arising in Clifford analysis, *Proc. R. Irish Acad., Section A.85*, pp. 1–23, 1985. (b) Clifford matrices, Cauchy–Kowalewski extension and analytic functionals, *Proc. Centre Math. Annal Aust. Natl. Univ.*, 16, pp. 284–299, 1988.

2.12.2 Study of Classical Möbius Transformations of $\mathbf{R}^n$

The concerned space is there the compactified $\mathbf{R}^n \cup \{\infty\}$ of $\mathbf{R}^n$. Some authors such as P. Lounesto use the following classical terminology.

2.12.2.1 Definition A Möbius transformation is called sense-preserving if $\det(df) > 0$, and sense-reserving if $\det(df) < 0$.

As already seen, the Möbius group of $\mathbf{R}^n \cup \{\infty\}$ has two components, the identity component being the sense-preserving Möbius group. We have already noticed that the full Möbius group of $\mathbf{R}^n \cup \{\infty\}$ is generated by translations, reflections, and the inversions $x \to x^{-1} = \frac{x}{x^2}$, or equivalently, by reflections in affine hyperplanes and inversions in spheres not necessarily centered at the origin. We have already noticed that the sense-preserving Möbius group is classically generated by the following four types of transformations: rotations, translations, positive dilatations, and transversions (or conformal special transformations). Transversions can be written

$$x \to \frac{x + x^2 c}{1 + 2B(x, c) + x^2 c^2}$$

with $c \in \mathbf{R}^n$ or in the equivalent forms $x \to (x^{-1} + c)^{-1}$ and $x \to x(cx + 1)^{-1}$. As emphasized by Pertti Lounesto,[163]

> *This might suggest the following: Let a, b, c, d in the Clifford algebra $\mathbf{C}_n$. If $(ax + b)(cx + d)^{-1}$ is in $\mathbf{R}^n$ for almost all $x \in \mathbf{R}^n$ and if the range of $g(x) = (ax + b)(cx + d)^{-1}$ is dense in $\mathbf{R}^n$, then g is a Möbius transformation of $\mathbf{R}^n$. Although this is true, the group so obtained is too large to be a practical covering group of the full Möbius group.*

Using Lounesto's notation for any a in $\mathbf{C}_n$, we put $\pi(a) = \hat{a}$, where π is the main automorphism of the considered Clifford algebra $\tau(a) = \tilde{u}$ and for $\nu = \pi \circ \tau = \tau \circ \pi$, $\nu(a) = \bar{a}$.[164] π is called by Pertti Lounesto the grade involution, τ is called the reversion, and ν the Clifford conjugation. One can easily verify that with previous notation for u belonging to the space called the space of p-vectors, we have $\pi(u) = (-1)^p u$ and $\tau(u) = (-1)^{(1/2)p(p-1)} u$.

The following definition has been given by H. Maass[165] and L. V. Ahlfors.[166] We denote there by Γ_n the Clifford group—also called the Lipschitz group.

[163] P. Lounesto, *Clifford Algebras and Spinors*, op. cit., p. 246.

[164] P. Lounesto, op. cit., p. 29 and p. 56.

[165] H. Maass, Automorphe Funktionen von mehreren Veränderlichen und Dirichletsche Reihen, *Abh. Math. Sem. Univ. Hamburg*, 16, pp., 1949.

[166] L. V. Ahlfors, Old and new in Möbius groups, *Ann. Acad. Sci. Fenn., serie A.1 Math.*, 9, pp. 93–105, 1984.

2.12.2.2 Definition The matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}_2(C_n)$ (the set of 2×2 matrices with entries in the Clifford algebra C_n) satisfying the conditions

(a) $a, b, c, d \in \Gamma_n \cup \{0\}$,
(b) $a\tilde{b}, \tilde{b}d, d\tilde{c}, \tilde{c}a \in \mathbf{R}^n$,
(c) $a\tilde{d} - b\tilde{c} \in \mathbf{R} \setminus \{0\}$,

is called a Vahlen matrix of the Möbius transformation g of $\mathbf{R}^n$ given by $g(x) = (ax + b)(cx + d)^{-1}$.

2.12.2.3 Some Results[167]

The Vahlen matrices form a group under matrix multiplication: the Vahlen group. The Vahlen group has a normalized subgroup in which condition (c) is replaced by (c′) $a\tilde{d} - b\tilde{c} = \pm 1$. The normalized Vahlen group is a fourfold, or rather double twofold, covering group of the full Möbius group of $\mathbf{R}^n$. The kernel of the covering homomorphism consists of

$$\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} e_{12\cdots n} & 0 \\ 0 & -\hat{e}_{12\cdots n} \end{pmatrix},$$

where $e_{12\cdots n}$ denotes the product of elements of the chosen basis of E_n. The sense-preserving Möbius group has a nontrivial twofold covering group formed by normalized Vahlen matrices with even diagonal (and odd off-diagonal) and pseudo-determinant $a\tilde{d} - b\tilde{c}$ equal to 1. The full Möbius group has a nontrivial twofold covering group with two components. The nonidentity component consists of normalized Vahlen matrices with odd diagonal (and even off-diagonal) and pseudodeterminant $a\tilde{d} - b\tilde{c}$ equal to -1.

2.12.3 Study of the Anti-Euclidean Case $E_{n-1}(0, n-1)$

We consider a $(n-1)$-dimensional anti-Euclidean space. For $x \in E_{n-1}(0, n-1)$, $q(x) = -(x_1^2 + \cdots + x_{n-1})^2$, where q denotes the quadratic form.

2.12.3.1 Definition By definition,[168] the sums of scalars and vectors are called paravectors. Paravectors span the linear space $\mathbf{R} \oplus E_{n-1}(0, n-1)$, denoted by $\$\mathbf{R}^n = \mathbf{R} \oplus E_{n-1}(0, n-1)$. As an extension of the Lipschitz group, I. R. Porteous[169] introduced the group of products of invertible paravectors $\$\Gamma_n$.

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[167] H. Maass, Automorphe Funktionen von mehreren Veränderlichen und Dirichletsche Reihen, *Abh. Math. Sem. Univ. Hamburg*, 16, pp. 72–100, 1949.
[168] P. Lounesto, op. cit., p. 247.
[169] I. R. Porteous, *Topological Geometry*, op. cit.

2.12.3.2 Proposition $\$\mathbf{R}^n$ *is isometric to the Euclidean space* $\mathbf{R}^n$.

Let $x = x_0 + x \in \mathbf{R} \oplus E_{n-1}(0, n-1)$, with $x_0 \in \mathbf{R}$ and $x \in E_{n-1}(0, n-1)$. Let us introduce the quadratic form

$$q_1(x) = x\bar{x} = x_0^2 - q(x) = x_0^2 + x_1^2 + \cdots + x_{n-1}^2,$$

from which the result can be deduced. K. Th. Vahlen[170] originally introduced the sense-preserving Möbius group of the paravector space $\$\mathbf{R}^n$.

2.12.3.3 Definition The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(C_{0,n-1})$ satisfying the conditions

(a) $a, b, c, d \in \$\Gamma_n \cup \{0\}$,
(b) $\bar{a}b, b\bar{d}, \bar{d}c, c\bar{a} \in \$\mathbf{R}^n$,
(c) $a\tilde{d} - b\tilde{c} = 1$,

is a Vahlen matrix with pseudodeterminant $a\tilde{d} - b\tilde{c} = 1$ of the sense-preserving Möbius transformation g of $\$\mathbf{R}^n$ given by $g(x) = (ax + b)(cx + d)^{-1}$.[171]

These Vahlen matrices with pseudodeterminant equal to 1 constitute a group that is a nontrivial twofold covering of the sense-preserving group of $\$\mathbf{R}^n$.

2.12.4 Study of Indefinite Quadratic Spaces

In addition to the above literature, we can give the following works: Elstrodt, Grunewald and Mennicke,[172] Fillmore and Springer,[173] Gilbert and Murray,[174] Hestenes and Sobczyk,[175] Lounesto and Springer.[176] As already shown, the full Möbius group of the compactification of $E_n(p, q)$ has two components (if either p or q is even), or four components (if both p and q are odd). With Lounesto's notation, $\Gamma_{p,q}$ stands for the Lipschitz (or Clifford) group and[177]

$$\mathrm{Spin}_+(p, q) = \{s \in \Gamma_{p,q} \cap C^+_{p,q} | s\tilde{s} = 1\}.$$

170 K. Th. Vahlen, op. cit.

171 K. Th. Vahlen, op. cit.

172 J. Elstrodt, F. Grunewald, and J. Mennicke, Vahlen's groups of Clifford matrices and spin groups, *Math. Z.*, 196, pp. 369–390, 1987.

173 J. P. Fillmore and A. Springer, Möbius groups over general fields using Clifford algebras associated with spheres, *Internat. J. Theoret. Phys.*, 29, pp. 225–246, 1990.

174 J. Gilbert and M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 34–38 and 278–296, 1991.

175 D. Hestenes and G. Sobczyk, *Clifford Algebras to Geometric Calculus*, D. Reidel, Dordrecht, the Netherlands, 1984, 1987.

176 P. Lounesto and A. Springer, Möbius transformations and Clifford algebras of Euclidean and anti-Euclidean spaces, in *Deformations of Mathematical Structures*, J. lawrynowicz, ed., Kluwer Academic, Dordrecht, pp. 79–90, 1989.

177 $\Gamma_{p,q} = \{s \in C_{p,q} : (\forall_{x \in E_n(p,q)}) s x \hat{s}^{-1} \in E_n(p, q)\}$, $\mathrm{Pin}(p, q) = \{s \in \Gamma_{p,q} / s\tilde{s} = \pm 1\}$, $\mathrm{Spin}(p, q) = \mathrm{Pin}(p, q) \cap C^+_{p,q}$ with P. Lounesto's notations (cf. P. Lounesto, op. cit., p. 220).

Any Möbius transformation $x \to (ax+b)(cx+d)^{-1}$ of $E_n(p,q)$, where $a, b, c, d \in C_{p,q}$, can be represented by a Vahlen matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathcal{M}_2(C_{p,q})$. More precisely, the entries a, b, c, d of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ are products of vectors and if invertible belong to the group $\Gamma_{p,q}$.

2.12.4.1 Proposition *The identity component of the Möbius group is generated by the rotations, translations, dilatations, and transversions, which are represented respectively as follows:*

$$
\begin{array}{lll}
axa^{-1}, & a \in \mathrm{Spin}_+(p,q), & \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \\
x+b, & b \in E_n(p,q), & \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \\
x\delta, & \delta > 0, & \begin{pmatrix} \sqrt{\delta} & 0 \\ 0 & \frac{1}{\sqrt{\delta}} \end{pmatrix}, \\
\dfrac{x + x^2c}{1 + 2B(x,c) + x^2c^2}, & c \in E_n(p,q), & \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}.
\end{array}
$$

The corresponding Vahlen matrices are given on the right.

2.12.4.2 Theorem (J. G. Maks[178]) *Let us consider four Vahlen matrices, which represent one rotation, one translation, one dilation, and one transversion. A product of these four matrices, in any order, has always an invertible entry in its diagonal (there are* $4! = 24$ *such products).*

Proof. For instance, in the product

$$
\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{\delta} & 0 \\ 0 & \frac{1}{\sqrt{\delta}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a\sqrt{b} + \frac{abc}{\sqrt{\delta}} & \frac{ab}{\sqrt{\delta}} \\ \frac{ac}{\sqrt{\delta}} & \frac{a}{\sqrt{\delta}} \end{pmatrix},
$$

the lower right-hand diagonal element $a/\sqrt{\delta}$ is invertible. To complete the proof of the fact that a product of rotation, a dilatation, and a transversion, in any order, is such that the Vahlen matrix representing it has always an invertible entry in its diagonal; one can verify the result in all the remaining 23 cases.

2.12.4.3 The Counterexample of J. G. Maks

In the general case ($p \neq 0$, $q \neq 0$), J. G. Maks[179] gave an example of a Vahlen matrix none of whose entries is invertible and all of which are nonzero. This example will be given in the exercises and proves that condition (a) has to be modified in the definition of a Vahlen matrix. Thus, P. Lounesto[180] introduces the closure $\pi_{p,q}$ of $\Gamma_{p,q}$, $\pi_{p,q}$ being the set of products of vectors, possibly isotropic of $E_n(p,q)$.

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179 J. G. Maks, op. cit., p. 41.

180 P. Lounesto, op. cit., p. 250.

2.12.4.3.1 Definition The matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}_2(C_{p,q})$ satisfying the conditions

(a) $a, b, c, d \in \pi_{p,q}$,
(b) $\bar{a}b, b\bar{d}, \bar{d}c, c\bar{a} \in E_n(p, q)$,
(c) $a\tilde{d} - b\tilde{c} \in \mathbf{R} \setminus \{0\}$,

is a Vahlen matrix of the Möbius transformation g of $E_n(p, q)$ given by $g(x) = (ax + b)(cx + d)^{-1}$.[181]

The Vahlen matrices here introduced form a group under matrix multiplication. The normalized Vahlen matrices, with pseudodeterminant satisfying $a\tilde{d} - b\tilde{c} = \pm 1$, form a fourfold, possibly trivial, covering group of the full Möbius group of $E_n(p, q)$. When both p and q are odd, the normalized Vahlen group is a nontrivial fourfold covering group of the full Möbius group of $E_n(p, q)$. When either p or q is even, one can find a nontrivial twofold covering group of the full Möbius group, consisting of the identity component of the normalized Vahlen group, that is, of normalized Vahlen matrices with even diagonal and pseudodeterminant equal to 1 and another component not containing the (nontrivial) preimages of the identity

$$\pm \begin{pmatrix} e_{12\cdots n} & 0 \\ 0 & \hat{e}_{12\cdots n} \end{pmatrix}.$$

The identity component of the normalized Vahlen group is a twofold (either p or q is even) or fourfold (both p and q are odd) covering group of the sense-preserving Möbius group.

2.12.4.3.2 The Counterexample of J. Cnops[182]

This counterexample will be given in the exercises.

2.13 Exercises

(I) Show Proposition 2.2.1.1.1. Hints: Take into account the definition of the angle θ between two intersecting real hypersphere, defined as the angle between hyperplanes tangent to hyperspheres at a common point.

(II) Show Proposition 2.2.1.1.2. Hints: Use an homogeneous equation of $\pi(Y)$.

(III) (We follow the method given by Ricardo Benedetti and Carlo Petronio in their book: Lectures on hyperbolic geometry, op. cit, pp. 10–22.) Show Proposition 2.3.3.

[181] J. P. Fillmore and A. Springer, op. cit.

[182] J. Cnops, Vahlen matrices for non-definite matrices, pp. 155–164 in R. Ablamowicz, P. Lounesto, J. M. Parra (eds.), *Clifford Algebras with Numeric and Symbolic Computations*, Birkhäuser, Boston, MA, 1996.

(1) **First case** $n = 2$

(a) Show that if M and N are connected oriented riemannian surfaces (naturally endowed with complex structures), the set of all conformal diffeomorphisms of M onto N is the set of all holomorphisms and all antiholomorphisms of M onto N.

(b) Show that the group $\mathrm{Conf}^+(S^2)$ consists of all homographies and the group $\mathrm{Conf}(S^2)$ consists of all homographies and antihomorgraphies, where $S^2 = \mathbf{C}P^1$ is naturally identified with the set $\mathbf{C} \cup \{\infty\}$ (where $\infty = 0^{-1}$) and a homography is a mapping of $\mathbf{C}P^1$ into itself such as $z \to (az+b)/(cz+d)$, and an antihomography is a mapping of $\mathbf{C}P^1$ into itself such as $z \to (a\bar{z}+b)/(c\bar{z}+d)$, where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ varies in $GL(2, \mathbf{C})$.

(c) Then, prove Theorem 2.3.3 for $n = 2$.

(2) **Second case** $n \geq 2$**.** U and V are domains in $\mathbf{R}^n$ and $f : U \to V$ is assumed to be a conformal diffeomorphism. We say that $f : x \to \lambda A i(x) + b$ is of type (a) if i is the identity, and of type (b) if i is the inversion with respect to a sphere.

(A) (a) Show that f is of type (a) if and only if ρ_f is constant (where $\rho_f = (\mu_f)^{-1}$) with $\|D_x f(v)\| = \mu_f(x)\|v\|$, for any $x \in U$, $v \in \mathbf{R}^n$.

(b) Show that f is of type (b) if and only if there exist $x_0 \in \mathbf{R}^n$ and $\eta \in \mathbf{R} \setminus \{0\}$ such that $\rho_f(x) = \eta\|x - x_0\|^2$.

(B) Show that there exist $\eta, r \in \mathbf{R}, z \in \mathbf{R}^n$ such that $\rho_f(x) = \eta\|x\|^2 + B(x, z) + r$.

(C) Show that if in the case (B), $\eta \neq 0$, then for some $x_0 \in \mathbf{R}^n$, we have $\rho_f(x) = \eta\|x - x_0\|^2$.

(D) Show that in the case (B), it cannot occur that $\eta = 0$ and $z \neq 0$. According to (B), we will say that f is of type (α) if $\eta = 0$ and $z = 0$ or of type (β) if $\eta = 0$ and $z \neq 0$, of type (γ) if $\eta \neq 0$. By (A), if f is of type (α), then it is of type (a) and (C) and (D) can be respectively be written as follows: If f is of type (γ) then it is of type (b). f cannot be of type (β).

(3) Prove the following Corollary: $\mathrm{Conf}(S^n)$ consists of all and only all the mappings of the form $x \to \mu B i(x) + w$, where $\mu > 0$, $B \in O(n)$, i is either the identity or the inversion with respect to a sphere and $w \in \mathbf{R}^n$.

(IV) Prove Theorem 2.4.1.1. Let (ε, η) be an isotropic base of $\mathbf{H}$, the standard hyperbolic plane $E_2(1,1)$ with $2B(\varepsilon, \eta) = 1$. Let $E = E_n(p, q)$.

(1) (a) Show that any element z in $F = E_n(p, q) \oplus \mathbf{H}$, $z = x + \alpha\varepsilon + \beta\eta$ belongs to $Q(F)$, the isotropic cone of F, if and only if $\alpha\beta = -q(x)$, where q is the quadratic form on $E_n(p, q)$, and that a vector $u = \alpha\varepsilon + x + \beta\eta$ belongs to the hyperplane tangent to $Q(F)$ along the generator line $\mathbf{R}z_0$, with $z_0 = \alpha_0\varepsilon + x + \beta_0\eta$, if and only if $\alpha\beta_0 + \alpha_0\beta + 2B(x, x_0) = 0$. We introduce the mapping $p : x \in E \to p(x) = \varepsilon + x - q(x)\eta$.

(b) Let V_0 be the intersection of $Q(F)$ with the affine hyperplane $\varepsilon + (E \oplus \mathbf{R}\eta)$ and $V = P(V_0)$ where P is the classical projection from F onto its associated projective space $P(F)$.

Show that $M = P(Q(F) \setminus \{0\})$ can be identified with the compactified space obtained by the adjunction to $E_n(p, q)$ of a projective cone at infinity.

(2) (a) Show that $PO(F) = O(F)/_{\{I,-I\}}$ acts on $P(F)$ and conserves $M = \tilde{Q}(F) = P(Q(F) \setminus \{0\})$ globally. $PO(F)$ is called the conformal group of E.

(b) Show that by passing to the projective space, $O(E)$ can be identified with a subgroup of $PO(F)$ of conformal automorphisms of M.

(c) Show that if a belongs to E, there exists $t_a \in O(F)$ such that for any x in E, $t_a(p(x)) = p(x+a)$ and that t_a leaves V_0 globally invariant and satisfies the relation $t_{a+b} = t_a \circ t_b$, for any a, b in E.

(d) Let λ be in $\mathbf{R}_+^*$. Show that we can associate with any positive dilation of E of coefficient λ an operator $t_\lambda \in O(F)$ such that $t_\lambda(p(x)) = \frac{1}{\lambda} p(\lambda x)$.

(e) Since the group of similarities $\mathcal{S}(E)$ of the affine space E is the product of three subgroups of $GL(E)$: $\mathcal{T}(E)$ the group of translations, $\mathcal{H}(E)$ the group of positive dilatations, and $\mathcal{O}(E)$; show that we can associate with any $s \in \mathcal{S}(E)$, t_s in $O(F)$, $t_s = t_\lambda t_a g$, with $g \in O(F)$ such that $g.\varepsilon = \varepsilon$ and $g.\eta = \eta$, with $\lambda \in \mathbf{R}_+^*$, $a \in E$. Let τ be the orthogonal symmetry relative to the unit vector $\epsilon + \eta$. Show that $\tau\epsilon = -\eta$ and $\tau\eta = -\epsilon$, and that τ leaves E invariant.

(f) Show that $s \to t_s$ is an isomorphism from $\mathcal{S}(E)$ onto the subgroup of elements of $O(F)$ which let the generator line $\mathbf{R}\eta$ of $Q(F)$ invariant together with an orientation of it, and that $s \to P \circ t_s$ is an isomorphism from $\mathcal{S}(E)$ onto the isotropy subgroup $S_{\tilde{\eta}}$ of the "point at infinity" $\tilde{\eta}$, in the group $PO(F)$.

(g) Conclude that M can be identified with $PO(F)/\mathcal{S}(E)$.

(h) Show that τ is the orthogonal symmetry relative to the unit vector $\varepsilon + \eta$. Show that $\tau\varepsilon = -\eta$ and $\tau\eta = -\varepsilon$. Show that $\tilde{\tau}$, the image of τ in $PO(F)$, corresponds in E to the classical inversion with center 0 and power 1. Using a theorem of J. Haantjes[183] that extends the theorem of Liouville (exercise III, above) to pseudo-Euclidean standard spaces $E_n(p,q)$, according to which the only conformal transformations of $E_n(p,q)$, $p+q \geq 3$, are products of affine similarities and inversions, conclude that $PO(F)$ is the group of all conformal diffeomorphisms of M.

(V) Prove Proposition 2.5.1.2 and 2.5.1.2.1, that is, if $n = 2r$, then $e_N f_{r+1}(-i)^{r-p} f_{r+1}$, where f_{r+1} is an $(r+1)$-isotropic vector and $f_{r+1} e_N = (-1)^{r+1}(-i)^{r-p} f_{r+1}$.

(VI) In 2.5.1.4 determine the connected components of $(S_c)_e$. Hints: Use the method and results given in 2.4.2.5 and also the results concerning the connected components of the classical spinoriality groups recalled in 3.10.

(VII) Prove 2.5.1.5. Hints: Use the method given in 2.5.1.4 and the results concerning the connected components of the classical spinoriality groups recalled in 3.10.

(VIII) (1) Prove Theorem 2.8.3.2 (structure equation):

(a) If X and Y are horizontal.

(b) If X and Y are vertical.

(c) If X is horizontal and Y vertical.

[183] Conformal representations of an n-dimensional Euclidean space with a nondefinite fundamental form on itself, *Nedel. Akad. Wetensch. Proc.*, 40, pp. 700–705, 1937.

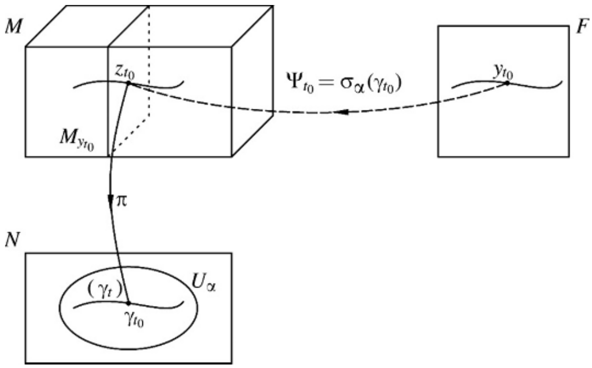
Conclude by using the following lemma (prove it): If A^* is the fundamental vector field corresponding to an element A of $Lie(G)$, and X is a horizontal vector field, then $[X, A^*]$ is horizontal.

(2) If both X and Y are horizontal vector fields on P, show that $w([X, Y]) = -2\Omega(X, Y)$.

(3) Prove Bianchi's identity (Theorem 2.8.3.3): $D\Omega = 0$.

(IX) cf. 2.8.7.14 Prove the following Proposition: Let E be a fiber bundle with typical fiber F, associated with a principal bundle P with structure group G, and let $H(E)$ be an Ehresmann connection on E. If G acts effectively on F and if $H(E)$ satisfies the following condition—For any differentiable path γ of the base B with origin point b_0 and for any element p_0 in the fiber of P in b_0, there exists a path (p_t) of P such that $\tau_\gamma^E \circ \tilde{p}_0 = \tilde{p}_t$, where τ_γ^E is the parallel displacement associated with the path γ—then there exists a unique principal connection $H(P)$ on P such that $H(E)$ is the connection associated with $H(P)$.

(X) A) Prove Proposition 2.8.8.4 using exercise IX.
B) Prove Proposition 2.8.8.5.



(1) (z_t) being horizontal, we have $z_t = \tau_\gamma(z_0)$, where τ_γ is the parallel displacement associated with the path γ_t. Let $t \to p_t$ be the horizontal lift of (γ_t) in P with origin ψ_0. Show that $\tau_\gamma = p_t \circ \psi_0^{-1}$ and $\psi_t = p_t.g_t$, where (g_t) is a path in G, and that the path (y_t) in F satisfies $\dot{y}_{t_0} = (\mu(-g_{t_0}^{-1}.\dot{g}_{t_0}))y_{t_0}$.

(2) Show that we have $\dot{\psi}_{t_0} = \dot{p}_{t_0}.g_{t_0} + p_{t_0}.\dot{g}_{t_0}$ and $w(\dot{\psi}_{t_0}) = (\sigma_\alpha^*(w)(\dot{\gamma}_{t_0})) = g_{t_0}^{-1}.\dot{g}_{t_0}$, whence the equation of the text results. Conclude.

(3) Prove the characterization given in 2.8.8.6.

(4) Prove the result given in Proposition 2.8.8.7.

(5) Find the local expression of example 2.8.8.8.

(6) Show that there is identity between the notion of $GL(F)$-Ehresmann connection and that of linear connection on a vector bundle (cf. below XII).

(7) Express the curvature of a $GL(F)$-Ehresmann connection on a vectorial bundle (M, π, N, F).

(8) Study the example 2.8.8.9, in detail:

(a) Give the local expression of $H(M)$.

(b) Determine the curvature $\tilde{\Omega}$ of the affine Ehresmann connection $H(M)$.

(9) Study the particular case of an affine Ehresmann connection on a vectorial fiber bundle.

(a) Prove the following result: Let i be the N-principal morphism from P into $P \times_{GL(E)} A(E)$ defined by $i(p) = \overline{(p, e)}$, $p \in P$. Then (i) and (ii) are equivalent:

(i) there exists a connection with the form $\tilde{w}$ on $P \times_{GL(E)} A(E)$.

(ii) there exists a connection with the form w and a 1-form θ with values in E, horizontal and of type $GL(E)$ on P.

(b) We assume now $\dim E = \dim N = n$. Let $H(M)$ be an affine Ehresmann connection on the vector bundle (M, π, N, E), w the form of connection on the bundle of frames P, and θ the 1-form on P, with values in E, horizontal and of type $GL(\bar{E})$ induced by $H(M)$ according to 9 (a). Let $p \to H_p(t) \subset T_p(P)$ be the field of horizontal subspaces induced by the connection of the form w on P. Show that (a) is equivalent to (b).

(a) For any element p in P, θ_p is an injective map (or surjective map) from $H_p(P)$ into E (and then bijective, taking account of the dimension).

(b) There exists a Cartan connection, called affine on P.

(XI) Justify the table of infinitesimal conformal transformations given in 2.9.1.3.3.

(XII) Study of Example 2.8.8.8. Let F be a real vector space of dimension m and let $\{e_i\}_{i=1,2,\ldots,m}$ be a fixed basis of F. Let (M, π, N, F) be a differentiable bundle with structure group $GL(F)$, provided with the structure of a vector bundle by using diffeomorphisms $\varphi_{\alpha,x}$ associated with a $GL(F)$-trivializing atlas for the transfer of the structure of a vector space F on the fibers. Let $H(M)$ be a $GL(F)$-Ehresmann connection on the vector bundle (M, π, N, F). $H(M)$ is associated with a connection with the form w with values in the Lie algebra $gl(F)$ on the principal associated bundle P, called the bundle of frames. Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ be the previous atlas.

(1) (a) Show that the mapping $\sigma_\alpha : x \to \sigma_\alpha(x) = \varphi_{\alpha,x}$ defines a local section over U_α of the principal bundle of frames.

(b) Show that we can associate with this section a basis of local sections of the bundle (M, π, N, F) defined by $x \to e_i(x) = \sigma_\alpha(x)e_i$, called the moving frame of the bundle (M, π, N, F).

(c) Let (w^i_j) be the components in the canonical basis (e^i_j) of $gl(F)$ of the local form $\sigma^*_\alpha(w)$, and $\{x^\lambda, y^i\}$ the system of local coordinates over $\pi^{-1}(U_\alpha)$, and let ϕ^i be the local 1-form on $\pi^{-1}(U_\alpha)$ defined by $\phi^i = \Gamma^i_\lambda dx^\lambda$, where Γ^i_λ are the local components of the Ehresmann connection $H(M)$ in the system of coordinates $\{x^\lambda, y^i\}$. Show that $\phi^i = \pi^*(w^i_j) \bullet y^j$.

(2) (a) Use α defined in Section 2.11.2.2 to prove the following result: the mapping ∇ is a linear connection on the bundle $((M, \pi, N, F)$ (cf. 2.8.7.4).

(b) Deduce that there is identity between the notion of $GL(F)$-Ehresmann connection and that of linear connection on a vector bundle.

(XIII) Links between conformal connections and riemannian connections in 2.9.2.2.2.4.3.

Notation: Let G be a subgroup of $G^r(n)$. We denote by $P^r(M)/G$ the quotient of $P^r(M)$ by the right action of G on $P^r(M)$, that is, the set of equivalence classes for the equivalence relation defined on $P^r(M)$ by "$j_0^r(f_2) \equiv j_0^r(f_1)$ if and only if there exists $a \in G$ such that $f_2 = f_1 \circ a$." We agree to denote by $G^r(n)/G$ the right quotient of $G^r(n)$ by G. $P^r(M)/G$ is a bundle associated with $P^r(M)$ with typical fiber $G^r(n)/G$. We will denote here by $\bar{\theta}$ the canonical form of $P^1(M)$, to distinguish it from that of $P^2(M)$; $\bar{\theta} = (\bar{\theta}^1, \ldots, \bar{\theta}^n)$ is an $\mathbf{R}^n$-valued 1-form such that in any system of coordinates (x^i), we have

$$\bar{\theta}^i\left(\sum_{k=1}^{n} X^k \frac{\partial}{\partial x^k}\right) = X^i.$$

If R_a denotes the right translation of $P^1(M)$, that is, $j_0^1(f) \to j_0^1(f \circ a)$, where $a \in GL(n, \mathbf{R})$, we have $R_a^*\bar{\theta} = ad(a^{-1})\bar{\theta} = a^{-1}\bar{\theta}a$. If $p : P^2(M) \to P^1(M)$ denotes the canonical projection $j_0^2(f) \to j_0^1(f)$ and if (θ^i, θ^i_j) denote the components of the canonical form of $P^2(M)$ in the canonical basis of $\mathbf{R}^n \oplus gl(n, \mathbf{R})$, we have $p^*\bar{\theta}^i = \theta^i$, $1 \le i \le n$. Let us recall (cf. 2.9.2.2.1.6) that a riemannian structure on M is a reduction to $O(n)$ of the structure group of $P^1(M)$. Then it is a subbundle $O_1(M)$ of $P^1(M)$ with structure group $O(n)$. We know also that the datum of such a reduction is equivalent to that of a cross section of the bundle $P^1(M)/O(n)$, (associated with $P^1(M)$, with typical fiber $GL(n, \mathbf{R})/O(n)$).

(1) Show that there exists a canonical bijective mapping between the subbundles of $P^1(M)$ with structure group $O(n)$ and the subbundles of $P^2(M)$ with structure group $O(n)$. Now we can assume that the riemannian structure of M is defined by a subbundle of $P^2(M)$ with structure group $O(n)$, which we agree to denote simply by $O(M)$. By restriction to $O(M)$ of the fundamental form θ, we obtain forms $(\bar{\theta}^i, \bar{\theta}^i_j)$ such that $\bar{\theta}^i_j$ is in the Lie algebra of $O(n)$ and thus skew-symmetric relative to the indices i and j. The forms $(\bar{\theta}^i, \bar{\theta}^i_j)$ are the forms of the riemannian connection of M and satisfy

$$d\bar{\theta}^i = -\sum_k \bar{\theta}^i_k \wedge \bar{\theta}^k,$$

and the curvature form $\bar{\Omega}^i_j$ of the riemannian connection is defined by

$$d\bar{\theta}^i_j = -\sum_k \bar{\theta}^i_k \wedge \bar{\theta}^k_j + \bar{\Omega}^i_j. \tag{1}$$

The forms $\bar{\Omega}^i_j$ vanish on the fibers, since the restrictions to the fibers of $O(M)$ of the forms $\bar{\theta}^i$, $\bar{\theta}^i_j$ can be identified with the Maurer–Cartan forms of $O(M)$. The method is the same as that used for the Ω^i_j and Ω_i. We can now set

$$\bar{\Omega}^i_j = \frac{1}{2}\sum_{k,h} R^i_{jkl}\bar{\theta}^k \wedge \bar{\theta}^h.$$

The tensor obtained is the curvature tensor of $O(M)$.

(2) Conformal structure associated with $O(M)$.

(a) Verify that according to Proposition 2.9.2.2.1.2 above, any riemannian structure $O(M)$ induces a conformal structure $P(M)$.

(b) Show that we have on $O(M)$ the relations $w^i = \bar{\theta}^i$, $w^i_j = \bar{\theta}^i_j$ with $w^i_j + w^j_i = 0$ and that the restrictions of the forms w_j to the fibers of $O(M)$ vanish.

(3) Prove the following result: two riemannian structures $O(M)$ and $O(M')$ determine the same conformal structure if and only if the riemannian associated "metrics" are conformal.

(4) Prove now the result given in 2.10.5.2: $x'(t)$ is an eigenvector of the corresponding Ricci tensor.

(XIV) Study of 2.11.2.2.3.2, 2.9.2.2.3.2, 2.9.2.2.3.3, 2.9.2.2.3.5. Prove the four given propositions.

(XV) Justification of Elie Cartan's presentation (2.10.2.1). It is enough to prove the existence of a unique local section σ of $P(M)$ such that the forms $\sigma^*\theta^i$ are given forms w^i, which constitute a basis of the cotangent space to M, and that with notation of 2.10.2,

$$\sigma^*\tilde{\theta}^0_0 = \sigma^*\theta^i_i = \frac{1}{n}\sum_{i=1}^{n}\sigma^*\theta^i_i = 0.$$

We start from any local section s such that $s^*\theta^i = w^i$, $1 \leq i \prec n$.

(1) As in Section 2.2, let us consider the Möbius group $\tilde{M}_n$, which acts on the Möbius space Q^n. Any element of $\tilde{M}_n$ corresponds to a couple of opposite matrices $(V, -V)$ in $O(q)$, with notation of 2.2, such that ${}^tVqV^{-1} = q$ (1), where q denotes, abusively, the matrix of the form q.

(a) Show that (1) can be expressed explicitly by the $(n+2)(n+3)/2$ relations

$$\sum_{r=1}^{n}\sum_{s=1}^{n} g_{rs}V_{ri}V_{sj} - V_{0i}V_{n+1,j} - V_{0j}V_{n+1,i} = q_{ij}, \tag{2}$$

with $q_{ij} = g_{ij}$, if $i, j \in \{1, 2, \ldots, n\}$, $q_{ij} = -1$ if $(i, j) = (0, n+1)$ or $(i, j) = (n+1, 0)$, and $q_{ij} = 0$ in all the other cases, that is, $i = 0$ or $n+1$ and $j \in \{1, 2, \ldots, n\}$, $j = 0$ or $n+1$, and $i \in \{1, 2, \ldots, n\}$, $(i, j) = (0, 0)$, $(i, j) = (n+1, n+1)$.

(b) Let $\tilde{M}_1(n)$ be the subset of $\tilde{M}(n)$, the Möbius group consisting of matrices V of the form

$$\begin{pmatrix} 1 & a_1 & \cdots & a_n & b^0 \\ 0 & & & & b^1 \\ \vdots & & \delta^i_j & & \vdots \\ 0 & \cdots & \cdots & \cdots & b^n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad \text{such that for } i = 1, 2, \ldots, n, \tag{3}$$

$$a_i = \sum_{j=1}^{n} g_{ij}b^j \quad \text{and} \quad b^0 = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}b^ib^j.$$

Show that $\tilde{M}_1(n)$ is a commutative and invariant subgroup of $O(q)$ isomorphic to $\mathbf{R}^n$.

(c) Show that with notation of 2.2, if $X = (X^0, \dots, X^{n+1})$, with the "point sphere" $\pi(X)$, the transformation associated with the matrix V associates the "point sphere" Y defined by

$$Y^i = \sum_{k=0}^{n+1} V_{ik} X_k.$$

(d) Since $g_{ij}X^iX^j - 2X^0X^{n+1} = g_{ij}Y^iY^j - 2Y^0Y^{n+1} = 0$, we can put $y_i = Y^i/Y^0$ for $1 \le i \le n$ and express the Cartesian coordinates y^i of the center of $\pi(Y)$ in terms of the coordinates $x^i = X^i/X^0$ of the center of $\pi(X)$ by

$$y^i = \frac{V_{i0} + \sum_k V_{ik}x^k + \frac{1}{2}V_{i,n+1}\sum_j\sum_k x^jx^k}{V_{00} + \sum_k V_{0k}x^k + \frac{1}{2}V_{0,n+1}\sum_j\sum_k x^jx^k} \quad (i, j, k = 1, \dots, n).$$

If $V_{00} \neq 0$, we can put $a^i = V_{i0}/V_{00}$, $a_i = V_{0i}/V_{00}$, $a^i_k = V_{ik}/V_{00}$ and thus determine y^i in terms of x^i, a_i, a^i_k, a^i.

(e) Since all the isotropy groups are isomorphic to one another, show that $\tilde{M}_\infty$ is the group of affine similarities of $\mathbf{R}^n$, which correspond to matrices V of the form

$$\begin{pmatrix} V_{00} & V_{01} & \cdots & V_{0,n+1} \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 0 & V_{n+1,n+1} \end{pmatrix}$$

with $V_{00}V_{n+1,n+1} = 1$,

$$\sum_{r,s=1}^{n} g_{rs}V_{ri}V_{sj} = g_{ij} \quad \text{for } i, j \in \{1, 2, \dots, n\},$$

$$\sum_{r,s} g_{rs}V_{ri}V_{s,n+1} = V_{n+1,n+1}V_{0i} \quad (1 \le i \le n),$$

$$\sum_{r,s} g_{rs}V_{r,n+1}V_{s,n+1} = 2V_{0,n+1}V_{n+1,n+1}.$$

(f) Show that for a matrix of type (3), the corresponding transformation is such that

$$y^i = \frac{x^i + \frac{1}{2}b^i\|x\|^2}{1 + \sum a_kx^k + \frac{1}{2}b^0\|x\|^2}, \quad \text{with } \|x\|^2 = \sum_{ij} g_{ij}x^ix^j.$$

(g) Show that any Möbius transformation can be written as the composite $s \circ \varphi$ of an affine similarity s and a transformation φ in $M_1(n)$.

(h) Show that the Möbius space Q^n can be identified with the homogeneous space $\tilde{M}(n)/\tilde{M}_0(n)$ or $\tilde{M}(n)/\tilde{M}_\infty(n)$, where $\tilde{M}_0(n)$ denotes the isotropy subgroup of the origin and $\tilde{M}_\infty(n)$ the isotropy subgroup of the point at infinity, ∞, and that $\tilde{M}_\infty(n)$ and $\tilde{M}_0(n)$ are of dimension $\frac{n^2+n+2}{2}$.

(i) Study the case of the canonical Euclidean structure $g_{ij} = \delta^i_j$.

(1) Determine the Lie algebras of $\tilde{M}(n)$ and $\tilde{M}_0(n)$.

(2) Using the results given in 1(b), show that if $s : U \to P(M)$ is a local given section, there exists a unique mapping $a : U \to M_1(n)$ such that

$$\sigma^*\bar{\theta}^0_0 = -\frac{1}{n}\sum_{i=1}^{n} \sigma^*\theta^i_i = 0.$$

A moving normal frame is a moving conformal frame A_p such that the forms (w^q_p) defined by

$$dA_p = \sum_{q=0}^{n+1} w^q_p A_q \quad \text{satisfy } w^0_0 = w^{n+1}_{n+1} = 0.$$

We agree to denote simply by $s = (A_p)$ the conformal moving frame associated with a section s of $P(M)$.

(3) Show that if $(w^i)_{1\le i\le n}$ is a system of differential forms over an open set U of M that constitute a moving tangent coframe, there exists a unique normal moving frame $\sigma = (A_p)$ such that $\sigma^*\theta^i = w^i$ $(1 \le i \le n)$, which is equivalent to

$$dA_0 = \sum_{i=1}^{n} w^i A_i.$$

The normal frame $\sigma = (A_p)$ is said to be associated with the tangent coframe (w^i).

(4) Let $\sigma = (A_p)$ be the normal moving frame associated with the coframe (w^i) over U and let ρ be a numerical nonvanishing function on U. The normal moving frame $\bar{\sigma} = (\bar{A}_p)$ associated with the coframe $(\bar{w}^i = \rho w^i)$ is then given by

$$\bar{A}_0 = \rho A_0,\ \bar{A}_i = A_i + \rho a_i A_0,\ \bar{A}_{n+1} = \rho\alpha A_0 + \sum_{i=1}^{n} a_i A_i + \frac{1}{\rho}A_{n+1},$$

where the functions a_i and α are respectively defined by

$$\frac{d\rho}{\rho^2} = \sum_{i=1}^{n} a_i w^i,\ \alpha = \frac{1}{2}\sum (a_i)^2.$$

$\bar{\sigma}$ is said to be deduced from σ by a dilation of the coefficient ρ.

(XVI) **2.10.4 Change of frames**

(1) Prove the following result: if γ is a path in M, there exists a normal moving frame (B_p) associated with γ such that the functions π^j_i of the differential system vanish completely.

(a) First study the case that a normal moving frame (B_p) is deduced from a normal moving frame (A_p) by the relation (1) of 2.10.4, and write the relations obtained and study the converse.

(b) Then deduce the result.

(XVII) Study of conformal normal connection. Let $\tilde{w}$ be the 1-form with values in the Lie algebra $po(p+1, q+1)$ of a conformal Cartan connection on a principal fiber bundle $(\tilde{P}, \tilde{\pi}, N, \tilde{G}_{n+1})$ with previous notation of 2.9.2.2.3. Then $\tilde{w}$ satisfies the following relations:

(i) $\tilde{w}(A^*) = A \; \forall \; A \in \mathcal{L}(\tilde{G}_{n+1}) = \tilde{g}_{n+1}$ (the Lie algebra of $\tilde{G}_{n+1}$).

(ii) $R_g^*.\tilde{w}(X) = adg^{-1}.\tilde{w}(X), \forall \; g \in \tilde{G}_{n+1}$.

(iii) $\tilde{w}(X) = 0$ is equivalent to $X = 0$.

Since the Lie algebra $po(p+1, q+1)$ is isomorphic to the Lie algebra $\mathbf{R}^n \oplus co(p,q) \oplus (\mathbf{R}^n)^* = (\mathbf{R}^n) \oplus \tilde{g}_{n+1}$, let $(\tilde{w}^i, \tilde{w}^i_j, \tilde{w}^0_i)$ be the components in the canonical basis $\{e_i, e^i_j, e^i\}$ of the Lie algebra $\mathbf{R}^n \oplus co(p,q) \oplus (\mathbf{R}^n)^*$ of the 1-form $\tilde{w}$. Notice that the form of components $(\tilde{w}^i_j, \tilde{w}^0_i)$ takes its values in the Lie algebra $\tilde{g}_{n+1}$.

(1) Give the components $(\tilde{\Omega}^i, \tilde{\Omega}^i_j, \tilde{\Omega}^0_j)$ in the base $\{e_i, e^i_j, e^i\}$ of the 2-form $\tilde{\Omega}$ of curvature of the Cartan connection defined by $\tilde{\Omega} = d\tilde{w} + \frac{1}{2}[\tilde{w}, \tilde{w}]$.

(2) Show that there exist functions $\tilde{K}^i_{kl}, \tilde{K}^i_{jkl}, \tilde{K}^0_{jkl}$ on $\tilde{P}$ such that $\tilde{\Omega}^i = \frac{1}{2} \cdot \tilde{K}^i_{kl}\tilde{w}^k \wedge \tilde{w}^l$, $\tilde{\Omega}^i_j = \frac{1}{2}\tilde{K}^i_{jkl}\tilde{w}^k \wedge \tilde{w}^l$, $\tilde{\Omega}^0_j = \frac{1}{2}\tilde{K}^0_{jkl}\tilde{w}^k \wedge \tilde{w}^l$.

A conformal Cartan connection on a principal bundle $\tilde{P}$ with structure group $\tilde{G}_{n+1}$ is called normal if its curvature form satisfies $\tilde{\Omega}^i = 0, i = , 2, \ldots, n$, $\tilde{\Omega}^i_j = (1/2) \tilde{K}^i_{jkl}\tilde{w}^k \wedge \tilde{w}^l$ with $\sum_i \tilde{K}^i_{jil} = 0$.

(3) We recall the following classical result (cf. for example Kobayashi S., *Transformation Groups in Differential Geometry*, op. cit., chapter IV):

Let $\tilde{P}$ a principal bundle with base N and structure group $\tilde{G}_{n+1}$ and let $(\tilde{w}^i, \tilde{w}^i_j)$ be a system of $(n+n^2)$ differential 1-forms on $\tilde{P}$ such that

(i) *$\tilde{w}^i(A^*) = 0$ and $\tilde{w}^i_j(A^*) = A^i_j$ for any fundamental vector field A^* generated by an element (A^i_j, A^0_j) of the Lie algebra $\tilde{g}_{n+1}$ of $\tilde{G}_{n+1}$.*

(ii) *$R_g^*(\tilde{w}^i, \tilde{w}^i_j) = adg^{-1}(\tilde{w}^i, \tilde{w}^i_j)$, for any $g \in \tilde{G}_{n+1}$.*

(iii) *Vertical vectors are those that satisfy $\tilde{w}^i(X) = 0$.*

(iv) *$d\tilde{w}^i + \tilde{w}^i_j \wedge \tilde{w}^j = 0$.*

Then, there exists a unique system of 1-forms $\{\tilde{w}^0_1, \ldots, \tilde{w}^0_n\}$ on $\tilde{P}$ such that $\{\tilde{w}^i, \tilde{w}^i_j, \tilde{w}^0_j\}$ define a conformal normal Cartan connection on $\tilde{P}$.

Prove the following result: We can canonically associate with any $O(p,q)$-structure $O(N)$ on N a normal Cartan connection on the principal bundle $\tilde{O}(N) = O(N) \times_{O(p,q)} \tilde{G}_{n+1}$.

(4) Study the case of the Cartan connection associated with a $CO(p,q)$-structure on N.

(XVIII) (1) Prove Theorem 2.10.5.3.

(2) Prove Theorem 2.10.5.5.

(3) Prove Theorem 2.11.2.1.2.

Hints.

- Use $\psi_{\alpha,x} : E_{n+2} \to (M_1)_x$, associated with the trivializing atlas $(U_\alpha, \psi_\alpha)_{\alpha \in A}$ of ζ_1 to determine bijective mappings $\bar{\psi}_{\alpha,x} : M_n \to \bar{M}_x$.
- Show that the mappings $(x, \bar{z}) \mapsto \bar{\psi}_{\alpha,x}(\bar{z})$ are bijective, that $\bar{\psi}_\alpha^{-1} \circ \psi_\beta$ are bijective and that $\psi_\alpha^{-1} \circ \psi_\beta$ are diffeomorphisms of $(U_\alpha \cap U_\beta) \times M_n$. Prove that there exists a unique structure of manifold over $\bar{M}$ such that $\xi = (\bar{M}, \bar{\pi}, N, M_n)$ is a C^∞ differentiable bundle. Prove that the principal bundle associated with ξ is isomorphic to bundle $\bar{P}$. Use the local chart of M_n defined in 2.9.1.3.1 and the mapping u defined in 2.4.2.2.1, to define a morphism of differentiable bundles $\bar{\varphi}$ as follows: (We recall that for any $y \in E_n$ we have $\hat{\varphi}^{-1}(y) = \overline{u(y)}$) let $y \in E_n, \in P_x$ and $p(y) = y_x$. Then we put $\bar{\phi}(y_x) = i(p)\overline{u(y)}$. $i(p)$ is identified with an element of the principal bundle associated with ζ, that is a diffeomorphism from M_n onto $\bar{M}_x$ —cf. footnote 154. Prove that $\bar{\varphi} : M \to \bar{M}$ sends a fiber into a fiber and therefore is an N–morphism of differentiable bundles and then a local diffeomorphism. Conclude.

(XIX) Study of 2.11.2.3.4. Let N be a pseudo-riemannian manifold provided with a "metric" g of type (p, q). Let $\xi = (T(N), \pi, N)$ and P be the principal bundle of orthonormal frames of N.

(1) First, show that the normal Cartan connection associated with the $O(p, q)$ structure on N defines a conformal generalized connection on the tangent bundle to N, which is called a normal conformal generalized connection.

(2) Let (e_i) be a moving orthonormal frame of ξ. We can canonically associate a local section S of P with (e_i). Show that the local 1-forms of the normal generalized connection are in the moving frame (e_i):

$$(\nu^i = S^* \bullet \theta^i;\ \nu^i_j = S^* \bullet w^i_j;\ \nu^0_j = S^* \bullet w^0_j),$$

where the (w^0_j) are the 1-forms on P defined by $w^0_j = i_1^*(\tilde{w}^0_j)$, with previous notation.

(3) (a) Show that the $\{\nu^i\}$ constitute the dual basis of the local basis of vector fields on N determined by the $\{e_i\}$.

(b) Show that the $\{\nu^i_j\}$ are the local components in the moving orthonormal frame $\{e_i\}$ of the Levi-Civita connection.

(c) Show that the $\{\nu^0_j\}$ are defined by

$$\nu^0_j = \left(\frac{-1}{n-2} R_{jl} + \frac{R}{2(n-1)(n-2)} g_{jl} \right) \nu^l,$$

where the $\{R_{jl}\}$ are the local components of the Ricci tensor in the moving frame $\{e_i\}$ and R is the riemannian scalar curvature and that the local curvature 2-forms (N^i_j) of the normal generalized connection satisfy $N^i_j = (1/2) A^i_{jkl} \nu^k \wedge \nu^l$, where the A^i_{jkl} are the local components in the moving frame $\{e_i\}$ of the Weyl tensor.

(XX) Prove Proposition 2.11.2.2.8. For the necessary condition use formulas (10) in 2.11.2.2.7. For the converse, use the covering (U_α) of N, where each U_α is the domain of definition of a moving frame of ξ, and over each U_α 1–forms satisfying the relations (12) and (14) over any $U_\alpha \cup U_\beta$. Conclude.

(XXI) 2.11.2.3.5.

(A) Equations of electromagnetic field in classical general relativity.
A. Lichnerowicz, in "Theories relativistes de la gravitation et l'electromagnetisme," op. cit., shows that in the case of a charged particle in an electromagnetic field such that the ratio of the charge to the mass is a constant $k = e/m$, the motion is subordinate to the equations

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jl}\frac{dx^j}{ds}\frac{dx^l}{ds} = kF^i_j\frac{dx^j}{ds},$$

where Γ'^i_{jk} are the Christoffel symbols of the riemannian connection ∇ on the manifold V_4 of general relativity and where the "metric" is of type $(1,3)$ and where $(x^i)_{i=1,2,3,4}$ constitute a system of local coordinates of V_4 and where F^i_j are relative to the skew-symmetric doubly covariant tensor F that defines the electromagnetic field and satisfies Maxwell's equations $\nabla_k F^{kj} = J^j$, where (J^j) denote the components of the electric current and $\frac{1}{2}\eta^{jkli}\nabla_j F_{kl} = 0$, where (η^{jkli}) denote the components of the 4-form element of volume of the *orientable* manifold V_4. It is shown that the last equation implies the local existence of a potential vector φ_i such that

$$F_{ij} = \partial_i\varphi_j - \partial_j\varphi_i.$$

We introduce the conformal generalized connection on $T(V_4)$ defined locally by

$$DX = dX^i + w^i_j X^j + w^0_j\left(\frac{1}{2}g^{ji}X^2 - X^j X^i\right)\cdot \partial_x i,$$

where $X = X\partial_x i$ with $w^l_j = \Gamma^i_{jk}dx^k$ and $w^0_j = -2F_{jk}dx^k$.

(1) Show that D is defined intrinsically.

(2) With the previous assumptions, show that the trajectory of the considered particle is the path $u \to x_u$ of V_4 that satisfies that $D_{x'_u}x'_u$ vanishes identically, $ds/du = k = e/m$, where s is the length of the path, and x'_u is the tangent vector to the path $x(u)$.

(B) Let S be a conservative dynamical system with holonomic complete constraints, with n degrees of freedom satisfying the hypothesis of Painlevé. Let V be the space of configuration and let $(q^i)_{i=1,2,\ldots,n}$ be a system of local coordinates of V. Let $L = T_2 + T_1 + T_0 + U$ be the Lagragian of the system.

(1) Show that in the system of coordinates $(q^i, \dot{q}^i)$ of $T(V)$ we have $L = (1/2)g_{ij}\dot{q}^i\dot{q}^j + b_i\dot{q}^i + T_0 + U$, with $\det((g_{ij})) \neq 0$.

(2) Since we assume that Painlevé's hypothesis are realized, L is such that $\partial L/\partial t = 0$ and then $T_2 - T_0 - U = h$, with $h \in \mathbf{R}$. We consider D, the generalized conformal connection on $T(V)$, as the space of states locally defined by

$$DX = dX^i + w^i_j X^j + w^0_j\left(\frac{1}{2}g_{ij}X^2 - X^i X^j\right)\partial_{q^i},$$

where $X = X^i \partial_{q^i}$ with $w^i_j = \Gamma'^i_{jk} dq^k$, where Γ'^i_{jk} are the Christoffel symbols associated with the "metric" $g' = \rho g$, where $\rho = 2(T_0 + U + h)$ (we assume that $T_0 + U + h > 0$), and $w^0_j = \Gamma^0_{jk} dq^k$ with $\Gamma^0_{jk} = 2(\partial_k h_j - \partial_j h_k)$.

(a) Show that the trajectories corresponding to the energy h are the paths $t \to x_t$ of V that satisfy $D_{x'_t}(\rho^{-1} x'_t) = 0$.

(b) Show that the corresponding time law is given by $ds/dt = \sqrt{\rho}$, where s is the corresponding length of the curve.

(c) Find again the classical Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0.$$

(3) Study the case that T_1 vanishes identically, where D is the Levi-Civita connection on V and where $w^0_j = 0$, and find the classical Maupertuis principle.

(XXII) The counterexample of J. G. Maks and the counterexample of J. Cnops.

(A) Consider the Minkowski space-time $E_4(3, 1)$ and its Clifford algebra generated by e_1, e_2, e_3, e_4 satisfying $(e_1)^2 = (e_2)^2 = (e_3)^2 = 1$, $(e_4)^2 = -1$. Consider the Vahlen matrix

$$W = \frac{1}{2}\begin{pmatrix} 1 - e_{14} & -e_1 + e_4 \\ e_1 + e_4 & 1 + e_{14} \end{pmatrix}$$

with entries in $C_{3,1}$.

(a) Justify the isomorphism $C_{3,1} \simeq M(4, \mathbf{R})$.

(b) Verify that all the entries of W are noninvertible.

(c) Verify that the matrix W is connected to the identity.

(d) Conclude that W is a Vahlen matrix where none of the entries are invertible and all are nonzero.

(B) Consider the Minkowski space-time $E_4(3, 1)$ and its Clifford algebra isomorphic to $\mathcal{M}(4, \mathbf{R})$ and consider the Vahlen matrix

$$C = \frac{1}{2}\begin{pmatrix} 1 + e_{14} & (e_1 + e_4)e_{23} \\ (-e_1 + e_4)e_{23} & 1 - e_{14} \end{pmatrix}.$$

It satisfies $a, b, c, d \in \pi_{3,1}$, $a\tilde{d} - b\tilde{c} = 1$, and $a\tilde{b}, \tilde{b}d, d\tilde{c}, \tilde{c}a = 0 \in E(3, 1)$, but even then $a\bar{b}, b\bar{d}, \bar{d}c, c\bar{a} \notin E(3, 1)$. The mapping $g_C(x) = (ax + b)(cx + d)^{-1}$ is conformal. If the matrix C is multiplied on either side by

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 + e_{1234} & 0 \\ 0 & 1 - e_{1234} \end{pmatrix},$$

then $B = CD = DC$ is such that $g_B(x) = g_C(x)$ for almost all $x \in E(3, 1)$. Furthermore, B satisfies $a\bar{b}, b\bar{d}, \bar{d}c, c\bar{a} \in E(3, 1)$. The matrices satisfying $a, b, c, d \in \pi_{3,1}$, $a\tilde{d} - b\tilde{c} = 1$, and $a\tilde{b}, \tilde{b}d, d\tilde{c}, \tilde{c}a \in E(3, 1)$ do not form a group, but only a set that is not closed under multiplication. This set generates a group that is the Vahlen group with norm 1 multiplied by the group consisting of the matrices

$$\begin{pmatrix} \cos\varphi + e_{1234}\sin\varphi & 0 \\ 0 & \cos\varphi - e_{1234}\sin\varphi \end{pmatrix}.$$

All these matrices are preimages of the identity Möbius transformation.

2.14 Bibliography

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3

Pseudounitary Conformal Spin Structures

The third chapter deals with pseudounitary spin geometry and pseudounitary conformal spin geometry. First we present pseudounitary conformal structures over a $2n$-dimensional almost complex paracompact manifold V and the corresponding projective quadrics $\tilde{H}_{p,q}$ associated with the standard pseudo-hermitian spaces $\mathbf{H}_{p,q}$. Then we develop a geometrical presentation of a compactification for pseudo-hermitian standard spaces in order to construct the pseudounitary conformal group of $\mathbf{H}_{p,q}$ denoted by $CU_n(p,q)$. We study the topology of the projective quadrics $\tilde{H}_{p,q}$ and the "generators" of such projective quadrics. We define the conformal symplectic group associated with a standard real symplectic space $(\mathbf{R}^{2r}, F)$, denoted by $CSp(2r, \mathbf{R})$, where F is the corresponding symplectic form such that $CU_n(p,q) = CSp(2r, \mathbf{R}) \cap C_{2n}(2p, 2q)$, with the notation of Chapter 2.

The Clifford algebra $Cl^{p,q}$ associated with $\mathbf{H}_{p,q}$ is defined. The corresponding spinor group Spin $U(p,q)$ and covering group $RU(p,q)$ are defined. A fundamental commutative diagram of Lie groups associated with $RU(p,q)$ is given: a characterization of $U(p,q)$ is given that gives another covering group $\triangle U(p,q)$ of $U(p,q)$. The space S of corresponding spinors is defined and provided with a pseudo-hermitian neutral scalar product. The embeddings of spinor groups and corresponding projective quadrics are revealed.

Then, by using the results of Chapter 2, conformal flat pseudounitary geometry is studied. Two fundamental diagrams associated with $CU_n(p,q)$ are given. We introduce and give a geometrical characterization of groups called pseudounitary conformal spinoriality groups.

The study of conformal pseudounitary spin structures over an almost complex $2n$-dimensional manifold V is now presented. The part played by the groups called pseudounitary conformal groups is emphasized.

Exercises are given.

3.1 Pseudounitary Conformal Structures[1]

3.1.1 Introduction

Let V be an almost complex $2n$-dimensional paracompact manifold. We know that any tangent space at V at a point $x : T_x$ inherits a pseudo-hermitian structure of type (p, q), $p + q = 2n$ by the datum of f, a pseudo-hermitian sesquilinear form of type (p, q). Such fields are differentially dependent on $x \in V$. We say that V is endowed with an almost pseudo-hermitian structure.

Any almost complex manifold inherits an almost pseudo-hermitian structure and an almost symplectic one. Over an almost pseudo-hermitian manifold, the set of normalized orthogonal bases suitable for the almost pseudo-hermitian structure constitutes a principal bundle with structure group $U(p, q)$. (So, any almost complex manifold has its principal associated bundle of bases reducible to $U(p, q)$.)

Conversely, as in Lichnerowicz[2] for the case of almost hermitian structures, one can show that if over a $2n$-dimensional manifold, there exists a real 2-form of rank $2n$ F, there exists an almost pseudo-hermitian structure such that F is the fundamental 2-form and V inherits an almost pseudo-hermitian structure (and then an almost complex structure).

3.1.2 Algebraic Characterization

At any point $x \in V$, the tangent space T_x is equipped with a sesquilinear hermitian form f that determines the pseudo-hermitian scalar product of type (p, q). T_x is thus isomorphic to a standard space $H_{p,q}$ of type[3] (p, q) with $p + q = 2n = n'$ ($H_{p,q} = (\mathbf{C}^{n'}, f)$, f sesquilinear pseudo-hermitian form of type (p, q)). Let $\mathbf{C}^{n'}$, $n' = p + q$ be equipped with f. We write $f(x, y) = R(x, y) + iI(x, y)$. We can verify that sesquilinearity implies that

$$R(ix, iy) = R(x, y);\ I(ix, iy) = I(x, y),$$

$$I(x, y) = R(x, iy) = -R(ix, y);\ \ R(x, y) = I(ix, y) = -I(x, iy).$$

If, moreover, we assume that f is hermitian, we find that $R(x, y) = R(y, x)$ and $I(x, y) = -I(y, x)$. We recall also that if f is hermitian (resp. skew-hermitian), then if is skew-hermitian (resp. hermitian).

[1] 3.1 up to 3.8 have been published: cf. Pierre Anglès, *Advances in Applied Clifford Algebras*, 14, no. 1, 1, pp. 1–33, 2004. The following sections constitute the matter of another paper: Pierre Anglès, Pseudounitary conformal spin structures, to appear in the Proceedings of ICCA7, the seventh International Conference on Clifford Algebras and their Applications, May 19–29, Université Paul Sabatier, Toulouse, France.

[2] A. Lichnerowicz, Théorie globale des connexions et des groupes d'holonomie, Edition Cremonese, Rome, 1962.

[3] R. Deheuvels, Formes quadratiques et groupes classiques, Presses Universitaires de France, Paris, 1980.

We know that there is identity between the datum of a complex vector space structure and that of a real vector space equipped with a linear operator J such that $J^2 = -Id$. It is classical that for an n-dimensional complex space E, E is identical (naturally isomorphic) to $({}_{\mathbf{R}}E, J)$, where ${}_{\mathbf{R}}E$ is the $2n$-dimensional real space obtained by reductions of scalars of E to real numbers and J is the complex conjugation, i.e., an $\mathbf{R}$-linear operator in E such that $J^2 = -Id$. (Cf. for example, R. Deheuvels, Formes quadratiques et groupes classiques, op. cit., pp. 210–211.)

We will use special results of C. Ehresmann given in the following references: a) Sur la Théorie des espaces fibrés, Coll. Int. du C.N.R.S., Top. Alg., Paris, 1947, pp. 3–35; b) Sur les variétés presque complexes, Proc. Int. Congr. Math. For a more recent discussion cf. also the following remarkable publication: P. Liebermann and C. M. Marle,. Geometrie Symplectique, Bases Théoriques de la Mécanique, t.1, U.E.R. Math., Paris 7 (1986), chapter 1. The proofs are given in the Appendix 3.13.2.

Let W be an n-dimensional complex space. We recall (cf. Chapter 1) that a pseudo-hermitian form on W is a mapping n from $W \times W$ into C such that *(i)* for any fixed $y \in W$ the mapping $x \mapsto \eta(x, y)$ is $\mathbf{C}$-linear, *(ii)* for all $x, y \in W$ we have $\eta(x, y) = \overline{\eta(y, x)}$ (the complex conjugate of $\eta(x, y)$). η is said to be nondegenerate if for any $x \in W, x \neq 0$, there exists $y \in W$ such that $\eta(x, y) \neq 0$. η is said to be hermitian if for any $x \neq 0$ we have $\eta(x, x) > 0$. If η is hermitian, then, automatically, it is nondegenerate.

Let η be a pseudo-hermitian form on W, and let G and Ω be $\mathbf{R}$-bilinear, real-valued forms on W defined by $G(x, y) = Re(\eta(x, y))$ and $\Omega(x, y) = -Im(\eta(x, y))$. Then G is symmetric and Ω is skew–symmetric, any of them is nondegenerate if and only if η is nondegenerate, and G is positive definite if and only if η is hermitian.

A real linear operator J on a real vector space V such that $J^2 = -Id$ is called a complex operator on V. V admits such an operator if and only if V is even-dimensional. In such a case V can be given a complex structure by defining multiplication by a complex number $z = a + bi$ $a, b \in \mathbf{R}$ as $(a + bi)x = ax + bJx$. If (V, Ω) is a symplectic space, where V is an even-dimensional real vector space and Ω a non-degenerate skew–symmetric bilinear form on V, then a complex operator J on V is said to be pseudo-adapted (resp. adapted) to Ω if there exists a pseudo-hermitian (resp. hermitian) form η on (V, J) such that $\Omega = -Im\,\eta$. It follows immediately from the definition that J is pseudo-adapted to Ω if and only if J is a symplectic isomorphism that is it satisfies the following condition: $\Omega(Jx, Jy) = \Omega(x, y), \forall x, y \in V$. If such a condition is satisfied, the form η defined for any $x, y \in V$ by $\eta(x, y) = G(x, y) - i\Omega(x, y)$, where $G(x, y) = \Omega(x, Jy)$, is pseudo-hermitian, the unique pseudo-hermitian form such that $\Omega = -Im\,\eta$. Using these facts we can give the following result the proof of which will be given in the Appendix 3.13.2.

3.1.2.1 Theorem. *Let (V, Ω) be a symplectic space and let J be a complex operator on V that is pseudo-adapted to Ω, with $\Omega = -Im\,\eta$ and $G(x, y) = -\Omega(x, Jy)$ as described above. The unitary group $U(V, \eta)$ satisfies then the following relation:*

$$U(V, \eta) = Sp(V, \Omega) \cap O(v, G).$$

In particular $U(p, q)$ is the set of all elements $u \in SO(2p, 2q)$ such that $u \circ J = J \circ u$, and also $U(p, q) = SO(2p, 2q) \cap Sp(2(p+q), \mathbf{R})$.

3.1.3 Some remarks about the Standard Group $U(p, q)$[4]

$SU(p, q)$ denotes the subgroup of elements of $U(p, q)$ with determinant equal to 1. $U(p, q)/SU(p, q)$ is isomorphic to $U(1)$, the center of $U(p, q)$, for $p + q = n > 1$ and the index ν—as usually defined as the dimension of maximal totally isotropic spaces such that $2\nu \leq n$—such that $\nu \geq 1$, consists of dilatations $x \to \lambda x$ such that $\lambda\bar{\lambda} = 1$ and $\lambda^n = 1$.[5] The center of $U(p, q)$ will be denoted by $U(1)$. I as used in the special case of $U(r)$ by S. Kobayashi, Differential Geometry of Complex Vector Bundles, p. 14, Proposition 4.21, Princeton University Press, 1987.

3.1.4 An Algebraic Recall

It is classical[6] that any pseudo-hermitian form h on a complex finite dimensional space E satisfies the following property (known as the law of inertia):

For any basis $\{e_1, \ldots, e_n\}$ that diagonalizes h, the number p of vectors e_j for which $h(e_j, e_j) > 0$, the number q of vectors e_k for which $h(e_k, e_k) < 0$, the number r of vectors e_l for which $h(e_l, e_l) = 0$, are independent of the basis. h is said to be of type (p, q).

3.1.5 Connectedness

We recall the following classical result: The groups $U(p, q)$ and $SU(p, q)$ are connected.

3.1.6 General Definitions

Let V be an almost complex $2n$-dimensional manifold and let us denote by T_x the tangent space at $x \in V$ to V and $H(T_x)$ the real space of hermitian forms on T_x.

3.1.6.1 Definition A pseudounitary conformal structure of type (p, q), $p \geq 0, q \geq 0$, $p + q = 2n = n'$, on V is the datum in any point x of a line C_x of $H(T_x)$ formed by the scalar multiples of a hermitian form of type (p, q) that satisfies the following local lifting axiom: "There exists an open covering $(V_i)_{i \in I}$ of V and on any V_i an analytic section $y \in V_i \to h^i_y \in H(T_y)$ such that $h^i_y \in C_y$ for all $y \in V$."

[4] Cf., for example, R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., pp. 227–284, and J. Dieudonné, (a) *La Géométrie des Groupes Classiques*, op. cit., (b) *On the Structure of Unitary Groups*, op. cit. (c) *On the Automorphisms of the Classical Groups*, op. cit., (d) *Sur les Groupes Classiques*, op. cit., pp. 63–84.

[5] J. Dieudonné, *Sur les Groupes Classiques*, op. cit., p. 69.

[6] R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., p. 232, for example.

3.1.6.2 Definition A conformal isometry from V onto V', both equipped with a pseudounitary structure of type (p, q), is an analytic diffeomorphism from V onto V' such that $\Phi(C_x) = C_{\Phi(x)}$ for any $x \in V$. We recall that any pseudo-hermitian form can be written as the difference of two hermitian positive-definite forms.

An almost pseudo-hermitian structure on V determines an associated conformal pseudo-hermitian structure of type (p, q). According to Deheuvels,[7] the set of hermitian positive forms over T_x is a convex cone P of $H(T_x)$, and the set of strictly positive forms over T_x is a convex cone $\overset{\circ}{P}$ of $H(T_x)$ and $H(T_x) = P - P \overset{df}{=} \{a - b : a, b \in P\}$.

3.2 Projective Quadric Associated with a Pseudo-Hermitian Standard Space $H_{p,q}$

Let $E = H_{p,q}$ be the standard pseudo-hermitian space $\mathbf{C}^{p+q}$ equipped with the classical pseudo-hermitian scalar product

$$f(x, y) = \sum_{i=1}^{p} x^i \bar{y}^i - \sum_{k=p+1}^{p+q} x^k \bar{y}^k,$$

the unitary group of which is called pseudounitary group of type (p, q) and denoted by $U(p, q)$. The affine space associated with E inherits an almost pseudo-hermitian manifold structure by defining the scalar product in the vector space $E_x = x + E$, of vectors with origin x, by translation of the pseudo-hermitian scalar product of E.

Let us introduce the hermitian semiquadratic form r associated with the pseudo-hermitian sesquilinear form f. We know that r defined for any $x \in E$ by $r(x) = f(x, x)$ is such that $r(\lambda x) = |\lambda|^2 r(x)$, for all $\lambda \in C$. Moreover,

$$r(x) = f(x, x) = \sum_{i=1}^{p} |x^i|^2 - \sum_{k=p+1}^{p+q} |x^k|^2.$$

The function r takes real values. We set $p + q = n'$.

Let us introduce the isotropic cone Q minus its origin, which constitutes a singular submanifold of $H_{p,q} = E$, defined by $x \in Q \Leftrightarrow r(x) = 0$. Indeed, at any point $y \neq 0$ of a generator line $\mathbf{C}x$ of Q, the affine hyperplane T_y tangent at y to Q is identical to the hyperplane $T = y^{\perp}$ with equation:

$$x^1 \bar{y}^1 + \cdots + x^p \bar{y}^p - x^{p+1} \bar{y}^{p+1} - \cdots - x^{p+1} \bar{y}^{p+1} = 0,$$

which is singular with radical $T^{\perp} \cap T = (y^{\perp})^{\perp} \cap T = \mathbf{C}y$.

Any affine subspace S_x, with origin x, complementary to the line $\mathbf{C}x$ in T_x is : $S_x = x + S$—translation of a complement S of $\mathbf{C}x$ in T. S is a regular space of type $(p - 1, q - 1)$. The natural map $u \in S \to u \bmod x \in T/\mathbf{C}x$ from S into the quotient space $T/\mathbf{C}x$ induces a natural hermitian quadratic form on $T/\mathbf{C}x$ and the

[7] R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., p. 230.

vector subspace S_x of vectors with origin x is a regular subspace of T_x equipped with a hermitian form of type $(p-1, q-1)$ isomorphic to $T/\mathbf{C}x$.

Let P be the classic projection from $E \setminus \{0\}$ into its associated projective space $P(E)$. We assume that $x_1 \neq 0$. We can take $x^2/x^1, \dots, x^{p+q}/x^1$ for coordinates at $\tilde{x} = P(x)$. Let $y = (y^1, \dots, y^{p+q}) \in H_{p,q}$. We can express dP_x as

$$dP_x(y) = \left(\frac{y^2x^1 - y^1x^2}{(x^1)^2}, \dots, \frac{y^{p+q}x^1 - y^1x^{p+q}}{(x^1)^2}\right).$$

We observe that the tangent vectors $\{y \text{ at } x\}$ and $\{(\lambda y) \text{ at } (\lambda x)\}$ have the same image, with $\ker dP_x = \mathbf{C}x$. dP establishes natural linear isomorphisms: $dP_{\lambda x}$ from $(T_{\lambda x/\mathbf{C}x})$, $\lambda \neq 0$, onto $T_{\tilde{x}}$, and $T_{\tilde{x}}$ is equipped with a pseudo-hermitian form of signature $(p-1, q-1)$.

3.2.1 Definition The projective quadric $\tilde{Q} = \tilde{Q}(H_{p,q})$—dim $\tilde{Q} = p+q-2$—is naturally equipped with a pseudounitary conformal structure of type $(p-1, q-1)$. By definition, such a quadric is called the projective quadric naturally associated with the hermitian space $H_{p,q}$. We agree to denote by $\tilde{H}_{p,q}$ the projective quadric associated with $H_{p,q}$.

3.2.2 Remark Let us introduce ${}_{\mathbf{R}}H_{p,q} = E_1$, the real vector space subordinate to $H_{p,q}$ and the isotropic cone minus its origin $C^1_{2n'}$ of E_1. Since $r(x) = 0$ for $x \in H_{p,q}$ is equivalent to $R(\xi, \xi) = 0$, we can identify the isotropic cone of $H_{p,q}$ with that of E_1, which has equation

$$\sum_{j=1}^{p}\{(\xi^j) + (\xi^{n'+j})^2\} - \sum_{j=p+1}^{p+q}\{(\xi^j)^2 + (\xi^{n'+j})^2\} = 0.$$

Introduce the natural projective space $P(E_1)$ associated with E_1 and the projective quadric $\tilde{Q}(E_1) = P(C^{1*}_{2n'})$ in $P(E_1)$. $\tilde{Q}(E_1)$ is naturally equipped with a pseudo-riemannian conformal structure of type $(2p-1, 2q-1)$. Such a quadric real realization of $\tilde{Q}(H_{p,q})$ can be associated with $H_{p,q}$.

3.3 Conformal Compactification of Pseudo-Hermitian Standard Spaces $H_{p,q}$, $p+q=n$

3.3.1 Introduction

Let $H = H_{1,1}$ be the complex hyperbolic space equipped with an isotropic basis (ε, η) such that, $2f(\varepsilon, \eta) = 1$ (f denotes the pseudo-hermitian form on H). The direct orthogonal sum $F = H_{p,q} \oplus H = H_{p,q} \oplus H_{1,1}$ is a pseudo-hermitian standard space of type $(p+1, q+1)$. Let us introduce the isotropic cone $Q(F)$, dim $Q(F) = n+1$ and the projective quadric $\tilde{Q} = P(Q(F)\setminus\{0\}) = M_1$ in the projective space $P(F)$ with dim $M_1 = n$. R denotes the real part of f.

Let us recall that $H_{p,q} = \mathbf{C}^{p+q}$ is identified with $\mathbf{R}^{2(p+q)}$ according to the previous process, $\mathbf{R}^{2(p+q)}$ equipped with the following basis: $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$,[8] an orthonormal basis adapted to the complex structure determined by the R-linear map J such that $J^2 = -\mathrm{Id}$. In the same way, we identify $H_{1,1}$ with $\mathbf{R}^4$ of type $(2, 2)$ with the basis $\{e_0, Je_0, e_{n+1}, Je_{n+1}\}$, an orthonormal adapted basis such that $e_0^2 = 1 = (Je_0)^2; e_{n+1}^2 = -1 = (Je_{n+1})^2$.

The datum of $z = \alpha\varepsilon + x + \beta\eta \in F = H_{p,q} \oplus H$ with $\alpha\beta \in \mathbf{C}$ and $x \in H_{p,q}$ is equivalent to that of $Z = ae_0 + be_{n+1} + cJe_0 + dJe_{n+1} + x$ with $Z \in \mathbf{R}^{2n+4}(2p+2, 2q+2)$, $X \in \mathbf{R}^{2n}(2p, 2q)$ and $a, b, c, d, \in \mathbf{R}$. Thus $z \in Q(F)$ is equivalent to $r(z) = 0$, i.e., $R(Z, Z) = 0$, i.e., $R(X, X) + a^2 - b^2 + c^2 - d^2 = 0$. Moreover, $R(X, X) = f(x, x) = r(x) = Q_{2p,2q}(X)$, where $Q_{2p,2q}$ denotes the quadratic form naturally associated with the real symmetric bilinear form R. Thus $z \in Q(F)$ iff Z belongs to the isotropic cone of $\mathbf{R}^{2n+4}(2p+2, 2q+2)$, i.e., iff $r(x) = Q(2p, 2q)(X) = b^2 - a^2 + d^2 - c^2$.

We can choose : $a = c = \frac{1}{2\sqrt{2}}(r(x) - 1)$ and $b = d = \frac{1}{2\sqrt{2}}(r(x) + 1)$ and introduce the map $p_1 : X \to p_1(X)$, where

$$p_1(X) = \frac{r(x)}{2\sqrt{2}} \underbrace{(e_0 + J(e_0) + e_{n+1} + J(e_{n+1}))}_{\delta_0} + X - \frac{1}{2\sqrt{2}} \underbrace{(e_0 + J(e_0) - e_{n+1} - J(e_{n+1}))}_{\mu_0}$$

id est we introduce the following map p_1 from E into F ($E = H_{p,q}$, $F = H_{p,q} \oplus H_{1,1}$):

$$p_1(x) = r(x)\delta_0' + x - \mu_0',$$

where $\delta_0' = \frac{\delta_0}{2\sqrt{2}}$ and $\mu_0' = \frac{\mu_0}{2\sqrt{2}}$ such that $f(\delta_0', \delta_0) = 0 = f(\mu_0', \mu_0)$ and $f(\delta_0', \mu_0') = \frac{1}{2}$. Moreover (δ_0', μ_0') constitutes an isotropic basis of $H_{1,1} = \mathbf{C}_{1,1}^2$.

3.3.1.1 Definition The projective quadric $M_1 = P((Q(F))^*)$, the image by P of $(Q(F))^* = Q(F)\backslash\{0\}$ into the corresponding projective space, is called by definition the conformal compactification of $H_{p,q}$.

We are now going to justify such a definition. Let $z = \alpha\delta_0' + x + \beta\mu_0'$ with $x \in H_{p,q}, \alpha, \beta \in \mathbf{C}$. Moreover $z \in Q(F)$ iff $f(z, z) = 0$, id est $\alpha\bar{\beta} + \bar{\alpha}\beta + 2r(x) = 0$. A vector $\mu = \alpha\delta_0' + x + \beta\mu_0'$ belongs to the tangent hyperplane at $Q(F)$ along the generator line $\mathbf{C}z_0$ with $z_0 = \alpha_0'\delta_0' + x_0 + \beta_0'\eta_0'$ iff $\mu \in z_0^{\perp}$, i.e., iff α and β satisfy the relation $\alpha\bar{\beta}_0 + \bar{\alpha}_0\beta + 2f(x, x_0) = 0$. Let us introduce V_0, the intersection of $Q(F)$ and the affine hyperplane (of F) $\mu_0' + (E \oplus \mathbf{C}\delta_0')$; y belongs to V_0 iff $y = r(x)\delta_0' + x - \mu_0'$. The map $p_1 : x \in E \to -\mu_0' + x + r(x)\delta_0'$ is one-to-one from E onto V_0, and

[8] It is easy to verify that there exists such a basis that satisfies $(e_1)^2 = (Je_1)^2 = 1, \ldots, (e_p)^2 = (Je_p)^2 = 1, (e_{p+1})^2 = (Je_{p+1})^2 = -1, \ldots, (e_{p+q})^2 = (Je_{p+q})^2 = -1$. Cf. R. Deheuvels, Formes quadratiques et groupes classiques, op. cit., pp. 184–191 and pp. 220–245.

determines a bijective map between E and the generator lines of $Q(F)$ that do not belong to the hyperplane $T_\infty = E \oplus \mathbf{C}\delta'_0$, thus a one-to-one map from E onto $P(V_0) = V$ in the projective space $P(F)$. V is an open set of the projective quadric M_1, and M_1 is topologically $\bar{V}$ (the closure of V) in $P(F)$. $W = M_1 \setminus V$ is the image in $P(F)$ of the intersection W_0 of $Q(F)$ with the hyperplane $T_\infty = E \oplus \mathbf{C}\delta'_0$. T_∞ is singular with radical $T_\infty^\perp \cap T_\infty = \mathbf{C}\delta'_0$, and so tangent to the cone $Q(F)$ along the isotropic line $\mathbf{C}\delta'_0$. W is a degenerate quadric of dimension $n-1$ in the projective hyperplane $\tilde{T}_\infty$: it is the projective cone formed by the projective lines with origin $\delta'_0 \in T_\infty$ resting against the regular projective quadric $\tilde{Q}(E)$ of dimension $n-2$ lying in the subspace $P(E)$ of $P(F)$. Indeed, $z = x + \lambda\delta'_0$, with $x \in E$, belongs to $Q(F)$ iff $f(z,z) = 0 = f(x,x)$ and $W_0 = Q(E) + \mathbf{C}\delta'_0$. So the conformal compactification M_1 of $E = H_{p,q}$ can be obtained by adjunction to E of a projective cone at infinity.

Let us determine D_{p_1} at $x \in E$. First, we note that for all $x, u \in E$, $r(x+u) = r(x) + r(u) + 2R(x,u)$ with previous notation, since $f(x,u) + f(u,x) = f(x,u) + \overline{f(x,u)} = 2R_e(f(x,u)) = 2R(x,u)$.

So, $p_1(x+u) - p_1(x) = u + 2R(x,u)\delta'_0 + r(u)\delta'_0$ and then: $(dP_1)_x(u) = u + 2R(x,u)\delta'_0$. $(dp_1)_x$ is a linear injective map and realizes a linear isomorphism from E_x onto $S_{p_1(x)}$ the tangent subspace at $p_1(x)$ to V_0.

$S_{p_1(x)}$ is a complementary subspace of the generator line $\mathbf{C}p_1(x)$ in the tangent hyperplane at $p_1(x)$ to the cone $Q(F)$.

Moreover, as δ'_0 is isotropic and orthogonal to E, $r((dp_1)_x(u)) = r(u)$. Thus, $(dp_1)_x$ realizes a pseudo-hermitian isometry from E onto S_x (conservation of the hermitian quadratic form). p_1 is a pseudo-hermitian isometry from the almost pseudo-hermitian manifold E onto its image $V_0 \subset Q(F)$. If we consider $P \circ p_1$, where P is the classical projection onto the projective space, $P \circ p_1$ is a pseudounitary conformal isometry from E onto V.

3.4 Pseudounitary Conformal Groups of Pseudo-Hermitian Standard Spaces $H_{p,q}$

Any element u of the pseudounitary group $U(F) = U(H_{p+1,q+1}) = U(p+1, q+1)$ globally conserves the isotropic cone $Q(F)$ interchanging the generator lines and mapping "isometrically" the tangent hyperplane at y to $Q(F)$ onto the tangent hyperplane at $u(y)$ to $Q(F)$. By passing to the quotient space $P(F)$, $U(F)$ operates on the projective space by its image $PU(F) = PU(p+1, q+1) = U(p+1, q+1)/Z_{n+2}$, where the center Z_{n+2} of $U(p+1, q+1)$ consists of the λI with $\lambda \in \mathbf{C}$ and $\lambda\bar{\lambda} = 1$ and will be denoted by $U(1)I$; $PU(F) = PU(p+, q+1)$ globally conserves the projective quadric $M_1 = \tilde{Q}(F)$ and respects its pseudounitary conformal structure.

3.4.1 Definition We call by definition $PU(F) = U(p+1, q+1)/(U(1)I)$ the pseudounitary conformal group of $E = H_{p,q}$.

The pseudounitary group $U(p,q) = U(H_{p,q})$ can be naturally identified with the subgroup of elements u of $U(F)$ such that $u(\delta'_0) = \delta'_0$ and $u(\mu'_0) = \mu'_0$. Thus

$U(E) \cap U(1)I = \{I\}$. If $u \in U(E)$,

$$p_1(u(x)) = -\mu'_0 + u(x) + r(u(x))\delta'_0 = u(-\mu'_0) + u(x) + r(u(x))\delta'_0$$
$$= u(-\mu'_0 + x + r(x)\delta'_0) = u(p_1(x)),$$

since $p_1 \circ u = u \circ p_1$, u globally conserves the image $p_1(E) = V_0 \subset Q(F)$, and the restriction of u to V_0 is an "isometry" of the almost pseudo-hermitian manifold V_0 onto itself. By passing the projective space, $U(E) = U(p,q)$ can be identified with a subgroup of $PU(F)$ consisting of conformal automorphisms of M_1. $U(E)$ globally conserves the "projective cone at infinity" W.

3.4.2 Translations of E

First, we remark that the group of isometries of the almost pseudo-hermitian manifold E, consisting of the translations $T(E)$, cannot appear as a subgroup of $U(F)$, since any operator of $T(E)$ different from zero changes the origin. On the other hand, when transferred by p_1 onto V_0, the translations "become" a natural subgroup $T(V_0)$ of $U(F)$.

To any vector a of E corresponds an element t_a of $U(F)$ such that $t_a(p_1(x)) = p_1(x+a) = p_1 \circ t_a(x)$.

3.4.2.1 Definition We set by definition $t_a(\mu'_0) = (t_a(p_1(0)) = p_1(a) = -\mu'_0 + a + r(a)\delta'_0$, $t_a(x) = x + 2R(x,a))\delta'_0$ for all $x \in E$, $t_a(\delta'_0) = \delta'_0$.

We can immediately verify that t_a respects the pseudo-hermitian scalar product of F; thus $t_a \in U(F)$. Moreover,

$$t_a(p_1(x)) = t_a(-\mu'_0 + x + r(x)\delta'_0) = -\mu'_0 + a + r(a)\delta'_0 + x + 2R(x,a)\delta'_0 r(x)\delta'_0$$
$$= -\mu'_0 + a + x + (r(x) + r(a) + 2R(x,a))\delta'_0$$
$$= -\mu'_0 + a + x + r(x+a)\delta'_0 = p_1(x+a) = p_1 \circ t_a(x).$$

t_a globally conserves V_0. Its trace on V_0 is the image by p_1 of the translation by a in E and $t_{a+b} = t_a \circ t_b$.

3.4.3 Dilatations of E and the Pseudounitary Group Sim $U(p,q)$

Let us consider now a dilatation $k_1 : x \to \lambda x$ of $E = H_{p,q}$. We assume that λ is a strictly positive real number. Such a dilatation is a pseudounitary conformal transformation of E. We associate with k_1 the following operation h_λ of $U(F)$:

3.4.3.1 Definition Let $k_1 : x \to \lambda x$ of $E = H_{p,q}$, with λ a strictly positive real number. Set $h_\lambda(\mu'_0) = (1/\lambda)\mu'_0$, $h_\lambda(x) = x$ for all $x \in E$ and $h_\lambda(\delta'_0) = \lambda\delta'_0$.

Since $r(\lambda x) = |\lambda|^2 r(x)$ and since λ is chosen to be a strictly positive real scalar, we obtain $r(\lambda x) = \lambda^2 r(x)$. Thus $p_1(\lambda_x) = \lambda h_\lambda(p_1(x))$, i.e., $p_1 \circ k_1(x) = \lambda h_\lambda(p_1(x))$, or equivalently, $h_\lambda \circ p_1 = \frac{1}{\lambda} p_1 \circ k_1$. V_0 is not transformed into itself by h_λ, but the image of $p_1(x)$ by h_λ belongs to the generator line $p_1(k_1(x))$, and h_λ determines a conformal isometry of M_1 that conserves V and W globally.

We know that the group of affine similarities $S(E_1)$, where $E_1 = {}_{\mathbf{R}}H_{p,q} = R^{2p+2q}(2p, 2q)$, is classically the product of its three subgroups, $T(E_1)$, $H(E_1)$ (dilatations $\xi \to \lambda\xi$ with $\lambda > 0$), and $O(2p, 2q)$, and that any element s of $S(E_1)$ can be written uniquely as $s = h_\lambda \circ t_a \circ u$, with $\lambda > 0$, $u \in (2p, 2q)$, $a \in E_1$ such that for all $y \in E_1$, $s(y) = \lambda(a + u(y))$. We introduce the following definition:

3.4.3.2 Definition We define an affine pseudounitary similarity of $E = H_{p,q}$ to be any transformation of $E : s = k_\lambda \circ t_a \circ u$, where $u \in U(p, q)$, $t_a \in T(E)$, k_λ a dilatation of E with λ a strictly positive real. We define the affine pseudogroup of similarities as the group denoted by $\operatorname{Sim} U(p, q)$ generated by such transformations of E.

Let us now consider $s \in \operatorname{Sim} U(p, q)$. We associate with s the following element $t_s \in U(F) : t_s = h_\lambda \circ t_a \circ u$, with previous notation. Since $p_1 \circ u = u \circ p_1$ and $t_a \circ p_1 = p_1 \circ t_a$,

$$\begin{aligned} t_s \circ p_1(x) &= h_\lambda \circ t_a \circ u[p_1(x)] = h_\lambda \circ t_a \circ p_1[u(x)] = h_\lambda \circ t_a \circ p_1[u(x) + a] \\ &= \frac{1}{\lambda} p_1[\lambda u(x) \lambda a] = \frac{1}{\lambda} p_1(s(x)), \qquad \text{since } h_\lambda \circ p_1 = \frac{1}{\lambda} p_1 \circ k_1. \end{aligned}$$

On the hyperplane $T_\infty = E \oplus \mathbf{C}\delta'_0$, $t_s(x + \beta\delta'_0) = u(x) + \lambda[\beta + 2R(u(x), a)]\delta'_0$ according to previous results. Thus, $t_s(T_\infty) \subset T_\infty$. Conversely, we can remark that the conditions for an element $v \in U(F)$, $v(T_\infty) \subset T_\infty$ and $v\delta'_0 \in \mathbf{C}\delta'_0$ are equivalent. Indeed, $\mathbf{C}\delta'_0 = \operatorname{rad} T_\infty$ and $T_\infty = (\delta'_0)^\perp$. The subgroup of $U(F)$ consisting of the elements v such that $v(T_\infty) \subset T_\infty$ is the isotropy group of the generator line $\mathbf{C}\delta'_0$. It contains $U(1)I$, with previous notation. Let v be an element of this subgroup. If $v(\delta'_0) = \mu\delta'_0$, with $\mu < 0$, then $u = -v$ is also in the subgroup and we have $u(\delta'_0) = -\lambda\delta'_0$, with $\lambda > 0$.

The conservation of the pseudo-hermitian scalar product implies that $\lambda = |\lambda| > 0$, $u(\mu'_0) = -(1/\lambda)\mu'_0 + a + \lambda r(a)\delta'_0$ with $a \in E$, and if $x \in E$, $u(x) = w(x) + 2\lambda R(u(x), a)\delta'_0$ with $w \in U(E)$. Thus, $u = t_s$ with $s = k_\lambda \circ t_a \circ w$, with $\lambda > 0$.

One can easily verify that $t_{s' \circ s} = t_{s'} \circ t_s$. The map $s \to t_s$ is therefore an isomorphism from $\operatorname{Sim} U(p, q)$ onto the subgroup consisting of the elements of $U(F)$ that conserves the generator line $\mathbf{C}\delta'_0$ of $Q(F)$. If we consider $P \circ t_s$, then $s \to P \circ t_s$ is an isomorphism from $\operatorname{Sim} U(p, q)$ onto the isotropy group $S_{\tilde{\delta}'_0}$ of the "point at infinity" $\tilde{\delta}'_0$ in the group $PU(F)$.

3.4.4 Algebraic Characterization

Moreover, the classical Witt theorem can be applied to pseudounitary geometry.[9] Consequently, $PU(F)$ is transitive on M_1.

[9] J. Dieudonné, (a) *La Géométrie des Groupes Classiques*, op. cit.; (b) On the structure of unitary groups. I, *Trans. Am. Math. Soc.*, 72, 1952, p. 367–385; II, *Amer. J. Math.*, 75, 1953, p. 665–678.

3.4.4.1 Theorem *The pseudounitary conformal compactification M_1 of $E = H_{p,q}$ is identical to the homogeneous space $PU(F)/\mathrm{Sim}\, U(p,q)$, the quotient space of the projective unitary group of $F : PU(F)$ by $\mathrm{Sim}\, U(p,q)$, the group of similarities of $H_{p,q}$.*

In order to describe the action of $PU(F)$ on M_1, it is enough, for a particular point m of M_1, to determine a transformation of $PU(F)$ that sends m onto δ'_0, the others being obtained using the elements of the isotropy group $S_{\tilde{\delta}'_0}$. Let us introduce v_0, the unitary symmetry of F relative to the unitary vector $\delta'_0 + \mu'_0$ (since $r(\delta'_0 + \mu'_0) = r(\delta'_0) + r(\mu'_0) + 2R(\delta'_0, \mu'_0) = 1$), $v_0(\delta'_0) = -\mu'_0$, $v_0(\mu'_0) = -\delta'_0$, while $v_0(x) = x$ for all $x \in E$. We determine the action of v_0 on a point $y = p_1(x) = -\mu'_0 + x + r(x)\delta'_0$, $v_0(p_1(x)) = \delta'_0 + x - r(x)\mu'_0$.

• If $r(x) \neq 0$, $r(x) \in \mathbf{R}$, then

$$v_0(p_1(x)) = r(x)\left[-\mu'_0 + \frac{x}{r(x)} + \frac{\delta'_0}{r(x)}\right].$$

Set $x' = x/r(x)$ such that $r(x') = 1/r(x)$ and

$$p_1(x') = -\mu'_0 + x' + r(x')\delta'_0 = -\mu'_0 + \frac{x}{r(x)} + \frac{1}{r(x)}\delta'_0.$$

We obtain

$$v_0(p_1(x)) = r(x)p_1(x').$$

• If $r(x) = 0$, $p_1(x)$ is sent by v_0 into the hyperplane at infinity T_∞. The action of $\tilde{v}_0 = P(v_0) \in PU(F)$ corresponds to the classical inversion with center at the origin and with power $+1$ that sends "at infinity" all the points of the isotropic cone of $E = H_{p,q}$.

We notice that the inversion is not a transformation from E onto itself, on account of the existence of singular points, while its "realization" in M_1 is a conformal isometry of M_1 without any singular point. We have just defined the inversion $I(0, +1)$ with center 0 and power that appears while considering $x' = +x/r(x)$ with $r(x') = 1/r(x)$.

In the same way, the inversion $I(0, -1)$ with center 0 and power -1 is $x \to x' = -x/r(x)$. Classically, for the real pseudoorthogonal case, according to a theorem of Haantjes[10] that extends to pseudo-Euclidean spaces of signature (r, s) with $r + s \geq 3$ the theorem of Liouville, the only real pseudo-Euclidean orthogonal conformal transformations are the products of affine similarities and inversions. Since $H_{p,q}$ is identical to $(\mathbf{C}^{p+q}, f)$ and to $(\mathbf{R}^{2(p+q)}, J)$ provided with a real bilinear symmetric form of type $(2p, 2q)$, according to the study of the corresponding pseudoorthogonal conformal group $\mathbf{C}_{2(p+q)}(2p, 2q)$,[11] there cannot be

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[10] J. Haantjes, Conformal representations of an n-dimensional Euclidean space with a non definite fundamental form on itself, *Nederl. Akad. Wetensch. Proc.* (1937), pp. 700–705.

[11] P. Anglès, (a) Construction de revêtements du groupe conforme d'un espace vectorial muni d'une métrique de type (p, q), *Annales de l'Institut Henri Poincaré, Section A.* vol. XXXIII

other transformations than the previous ones in the pseudounitary conformal group of $H_{p,q}$.

Thus, we have obtained the following statement:

3.4.4.2 Proposition *The conformal pseudounitary group of* $H_{p,q} = E$ *is the group consisting of products of conformal pseudounitary similarities and inversions of* E.

3.5 The Real Conformal Symplectic Group and the Pseudounitary Conformal Group

3.5.1 Definition of the Real Conformal Symplectic Group

Let $(\mathbf{R}^{2r}, F)$ be a real symplectic standard space, where F is the standard sympletic form on $\mathbf{R}^{2r}$ defined as

$$F(x, y) = \sum_{j=1}^{r} (x^j y^{j+r} - x^{j+r} y^j).$$

We will denote by J the standard complex structure, pseudo-adapted to F—cf. Appendix 3.13.2. We call real conformal symplectic group and we denote by $CSp(2r, \mathbf{R})$, the group of transformations constituted by the linear symplectic automorphisms of $\mathbf{R}^{2r}$—(elements of $Sp(2r, \mathbf{R})$)—the translations and the dilatations of $(\mathbf{R}^{2r}, F)$.

As in the orthogonal case, we give the following definition:

3.5.1.1 Definition A continuously differentiable function f from an open set U of $(\mathbf{R}^{2r}, F)$ into U is conformal in U if and only if there exists a continuous function γ_1 from U into $\mathbf{R}^*$ such that for any $x \in U$ and for any $a, b \in \mathbf{R}^{2r}$ we have

$$F((D_x f)a, (D_x f)b) = \gamma_1(x) F(a, b),$$

where $D_x f$ is the linear mapping tangent to f at x.

3.5.1.2 Definition Let $\mathbf{H}_{p,q}$ be the standard pseudounitary space. Let f be the corresponding pseudo-hermitian scalar product. A continuously differentiable function φ from an open set U of $\mathbf{H}_{p,q}$ into $\mathbf{H}_{p,q}$ is pseudounitary in U if and only if there exists a continuous function λ from U into $\mathbf{C}^*$ such that for almost all $z \in U$ and for any $a, b \in U$ we have

$$f((D_z\varphi)a, (D_z\varphi)b) = |\lambda(z)|^2 f(a, b),$$

no 1, pp. 33–51, 1980; (b) Géométrie spinorielle conforme orthogonale triviale et groupes de spinorialité conformes, Report HTKK Mat A 195, Helsinki University of Technology, pp., 1982; (c) Real conformal spin structures, *Scientiarum Mathematicarum Hungarica*, vol. 23, pp. 115–139, Budapest, Hungary, 1988.

where $D_z\varphi$ denotes the linear mapping tangent to φ at z. The set of such transformations of $\mathbf{H}_{p,q}$ is denoted by $CU_n(p,q)$ and constitutes a group under the usual law of composition.

3.5.1.3 Theorem *$CU_n(p,q)$ is identical to the previous group $U(p+1,q+1)/U(1).I$.*

The proof will be given below in the exercises.

3.5.1.4 Theorem *$CU_n(p,q)$ is the set of $u \in C_{2n}(2p,2q)$, the real conformal orthogonal group in dimension $2n$ and signature $(2p,2q)$ such that Du, the linear tangent mapping to u, satisfies $Du \circ J = J \circ Du$. $CU_n(p,q)$ is the set of $u \in CSp(2n,\mathbf{R})$—the real conformal symplectic group—such that Du, the linear tangent mapping to u, satisfies $Du \circ J = J \circ Du$. $CU_n(p,q) = CSp(2n,\mathbf{R}) \cap C_{2n}(2p,2q)$.*

Proof. Let us put $\lambda(z) = \lambda_1(z) + i\lambda_2(z)$, where $i^2 = -1$. Thus, $|\lambda(z)|^2 = \lambda_1^2(z) + \lambda_2^2(z)$. Thus u belongs to $CU_n(p,q)$ if and only if for almost all $z \in \mathbf{H}_{p,q}$, for any $a, b \in \mathbf{H}_{p,q}$,

$$f((D_zu)a,(D_zu)b) = (\lambda_1^2(z) + \lambda_2^2(z))f(a,b).$$

Now,

$$\begin{aligned} f(z,z') &= \mathrm{Re}\, f(z,z') + i\,\mathrm{Im}\, f(z,z') = -F(Jz,z') - iF(z,z') \\ &= B(z,z') - iF(z,z'), \end{aligned}$$

where B denotes the bilinear symmetric form associated with f, and F is the corresponding symplectic form (cf. 3.1.2.1). First, we want to prove that if u belongs to $CU_n(p,q)$, then u belongs to $CSp(2n\mathbf{R}) \cap C_{2n}(2p,2q)$ and that $Du \circ J = J \circ Du$. Let $B(x,y) = F(x,Jy)$, and $f(x,y) = B(x,y) - iF(x,y)$ (see 3.1.2.1 and 3.13.6.2.6). Then it is immediate that if $u \in CU_n(p,q)$ implies $u \in CSp(2n\mathbf{R}) \cap C_{2n}(2p,2q)$. We will now prove that Du commutes with J. Notice that by definition $B((D_zu)a, (D_zu)b) = F((D_zu),a,J(D_zu)b)$. On the other hand, since $u \in CSp(2n\mathbf{R})$ with the same factor λ, we have that $B((D_zu)a,(D_zu)b) = |\lambda(z)|^2 B(a,b) = |\lambda(z)|^2 F(a,Jb) = F((D_zu)a,(D_z)uJb)$. So, we find that $F((D_zu)a, J(D_zu)b) = F((D_zu)a,(D_zu)Jb)$ and since F is nondegenerate we get $J \circ Du = Du \circ J$. The converse can be proven in much the same way.

3.6 Topology of the Projective Quadrics $\tilde{H}_{p,q}$

3.6.1 Topological Properties

Let $\{e_1,\ldots,e_{p+q}\}$ be an orthogonal normalized basis that diagnonalizes the scalar classical pseudo-hermitian product of $\mathbf{H}_{p,q}$, $n = p+q$. We denote by $\mathbf{H}$ the isotropic cone of $E = \mathbf{H}_{p,q}$, with equation

$$\sum_{j=1}^{p}\{(\xi^j)^2 + (\xi^{n+j})^2\} - \sum_{j=p+1}^{p+q}\{(\xi^j) + (\xi^{n+j})^2\} = 0,$$

where $x = \sum_{k=1}^{n} x^k e_k$ belonging to $\mathbf{C}^n$ is identified with $\xi = \sum_{k=1}^{n}(\xi^k + i\xi^{n+k})$ belonging to $\mathbf{R}^{2n}$, and $y = \sum_{k=1}^{n} y^k e_k$ is identified with $\eta = \sum_{k=1}^{n}(\eta^k + i\eta^{n+k})$. Let us introduce ${}_{\mathbf{R}}\mathbf{H}_{p,q} = E_1$, the real space subordinate to $\mathbf{H}_{p,q}$, and the classical quadric (hyperboloid) S the equation of which is

$$\sum_{j=1}^{p}\{(\xi^j)^2 + (\xi^{n+j})^2\} - \sum_{j=p+1}^{p+q}\{(\xi^j)^2 + (\xi^{n+j})^2\} = 2.$$

Thus $x \in S \cap \mathbf{H}$ iff

$$\sum_{j=1}^{p}\{(\xi^j)^2 + (\xi^{n+j})^2\} = 1 = \sum_{j=p+1}^{p+q}\{(\xi^j)^2 + (\xi^{n+j})^2\},$$

i.e., iff x belongs to the product of the unitary sphere $\sum_p$ of the standard hermitian space H_p by the unitary sphere $\sum_q$ of the standard hermitian space H_q. $\sum_p$ is classically isomorphic to S^{2p-1} and $\sum_q$ to S^{2q-1}. Let y be a point of $\mathbf{H} \setminus \{0\}$. Necessarily we have

$$\sum_{j=1}^{p}\{(\eta^j)^2 + (\eta^{n+j})^2\} = \sum_{j=p+1}^{p+q}\{(\eta^j)^2 + (\eta^{n+j})^2\} = \rho > 0.$$

The generator line $\mathbf{C}y$ is such that $\mathbf{C}y \cap (\sum_p \times \sum_q) = \{(e^{i\varphi}/\sqrt{\rho})y, \varphi \in \mathbf{R}\}$. Conversely, any $(a, b) \in \sum_p \times \sum_q$ belongs to a generator line of H that it determines. We have a natural mapping from $\sum_p \times \sum_q$ (or from $S^{2p-1} \times S^{2q-1}$) onto the projective quadric $\tilde{\mathbf{H}}_{p,q} = P(\mathbf{H}\setminus\{0\})$, where P is the standard projection from $\mathbf{H}_{p,q}$ onto its projective space, which enables us to identify $\tilde{\mathbf{H}}_{p,q}$ with the quotient of the manifold $S^{2p-1} \times S^{2q-1}$ by the equivalence relation $(a, b) \sim e^{i\varphi}(a, b)$ and thus realizes a $U(1)$ covering $(\sum_p \times \sum_q) = S^{2p-1} \times S^{2q-1}$ of $\tilde{\mathbf{H}}_{p,q}$.

Consequently, $\tilde{\mathbf{H}}_{p,q}$ is homeomorphic to $S^{2p-1} \times S^{2q-1}/S^1$. We recall that S^{2p-1} is a bundle over $P^{p-1}(\mathbf{C})$ with typical fiber S^1. It is one of the Hopf classical fibrations.[12] In fact, $P^{p-1}(\mathbf{C})$ is diffeomorphic to $U(p)/U(p-1) \times U(1)$ and homeomorphic to S^{2p-1}/S^1. Thus $\tilde{\mathbf{H}}_{p,q}$ is homeomorphic to $P^{p-1}(\mathbf{C}) \times S^{2q-1}$ and $S^{2p-1} \times P^{q-1}(\mathbf{C})$. Since $p > 1, q > 1$, we find again, since $P^{p-1}(\mathbf{C})$ is then simply connected[13] and since S^{2q-1} is simply connected, that $\tilde{\mathbf{H}}_{p,q}$ is simply connected, for $p > 1, q > 1$.[14]

[12] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, New Jersey, pp. 106, 107, 1951; A. L. Besse, *Manifolds All of Whose Geodesics Are Closed*, Springer-Verlag, New York, p. 75, 1978; D. Husemoller, *Fibre Bundles*, 3rd edition, McGraw Hill Book Company, New York, 1993; I. R. Porteous, *Topological Geometry*, 2nd edition, Cambridge University Press, 1981.

[13] A. L. Besse, op. cit., p. 83.

[14] Nordon J., Les éléments d'homologie des quadriques et des hyperquadriques, *Bulletin de la Société Mathématique de France*, tome 74, p. 124, 1946.

3.6.2 Generators of the Projective Quadrics $\tilde{H}_{p,q}$

As for the pseudo-Euclidean case, the maximal totally isotropic subspaces of $\mathbf{H}_{p,q}$ contained in the cone $\mathbf{H}$ have complex dimension equal to $\min(p, q)$. Their images in the projective space are the projective subspaces included in the projective quadric $\tilde{\mathbf{H}}_{p,q} = P(\mathbf{H}\backslash\{0\})$, which we agree to call generators of $\tilde{\mathbf{H}}_{p,q}$ of complex dimension $\inf(p, q) - 1$. Let us assume $p \geq q$. Let E_+ be the hermitian subspace of E with basis $\{e_1, \ldots, e_p\}$ and let E_- be the anti-hermitian subspace of E with basis $\{e_{p+1}, \ldots, e_{p+q}\}$ $E = E_+ \oplus E_-$. Any maximal totally isotropic subspace V of E determines canonically an anti-isometry φ_v from E_- into E_+: for all $t, t' \in E_-$,

$$f(\varphi_v(t), \varphi_v(t')) = -f(t, t').$$

Both V and E_- are complementary subspaces in E of E_+: $E = E_+ \oplus E_-$, $E = E_+ \oplus V$. If p_+ and p_- denote the restrictions to V of the projections onto E_+ and E_- of the first decomposition, p_- is a linear isomorphism from V onto E_-. We take $\varphi_v = p_+ p_-^{-1}$. If $t \in E_-$, then $p_-(t) = t + p_+ p_-^{-1}(t) = t + \varphi_v(t)$ belongs to V. For all $t, t' \in E_-$, $f(t + \varphi_v(t), t'\varphi_v(t')) = 0$ and φ_v is an anti-isometry. We can associate with V the orthogonal system $U = \Phi(V) = \{u_1 = \varphi_v(e_{p+1}), \ldots, u_p = \varphi_v(e_{p+q})\}$ of q vectors of E_+.

Conversely, with any orthogonal system U of q vectors $\{u_1, u_2, \ldots, u_q\}$ of E_+ we associate $V = \Psi(U)$ generated by the vectors $v = u_1 + e_{p+1}, v_2 = u_2 + e_{p+2}, \ldots, v_q = u_q + e_{p+q}$. The vectors $v_1, \ldots, v_q$ are linearly independent, isotropic, mutually orthogonal. V is then a maximal totally isotropic subspace. Φ and Ψ are inverse mappings that determine a natural one-to-one mapping between the set of maximal totally isotropic subspaces of E, or equivalently, the set of "generators" of the projective quadric $\tilde{\mathbf{H}}_{p,q}$, and the Stiefel manifold $V_{p,q}$ of systems of q orthogonal vectors of the hermitian space H_p. If $p > q$, such a manifold is identical to the quotient[15]

$$U(p)/U(p - q) = SU(p)/SU(p - q)$$

and is connected and simply connected. If $p = q$, Φ establishes a one-to-one mapping from the set of generators of $\tilde{\mathbf{H}}_{p,q}$ onto the set $V_{p,q}$ of orthogonal bases of $\mathbf{H}_{p,q}$, which is identical to the unitary connected group $U(p)$, not simply connected, with fundamental group classically isomorphic to Z.

3.7 Clifford Algebras and Clifford Groups of Standard Pseudo-Hermitian Spaces $\mathbf{H}_{p,q}$

3.7.1 Fundamental Algebraic Properties

We recall that $U(p, q)$ is the set of elements $u \in SO(2p, 2q)$ such that $u \circ J = J \circ u$ (J: transfer operator of the complex structure). Let us introduce $C_{2p,2q}$, the real Clifford algebra of $E(2p, 2q)$, the real pseudo-Euclidean standard space equipped with

[15] Husemoller D., Fibre bundles, Third edition, McGraw Hill Book Company, New York, 1993, p. 89.

a quadratic form of signature $(2p, 2q)$. $C_{2p,2q} = C^+_{2p,2q} \oplus C^-_{2p,2q}$ ($C^+_{2p,2q}$ even Clifford algebra and $C^-_{2p,2q}$= subspace of odd elements). $C^-_{2p,2q}$ can be seen as a $C^+_{2p,2q}$ module. We recall that $\mathbf{H}_{p,q}$ is identical to $(E(2p, 2q), J)$.

3.7.1.1 Theorem *There exists a linear mapping $\tilde{J}$ from $C_{2p,2q}$ into $C_{2p,2q}$ such that*

(a) $C^+_{2p,2q}$ *and* $C^-_{2p,2q}$ *are conserved by the action of* $\tilde{J}$*,*

(b) $\tilde{J}^2(c) = c, \forall c \in C^+_{2p,2q}$ *and* $\tilde{J}^2(c) = -c, \forall c \in C^-_{2p,2q}$*,*

(c) $\tilde{J}(c_1c_2) = \tilde{J}(c_1)\tilde{J}(c_2)$*, for all* $c_1, c_2 \in C_{2p,2q}$*.*

We consider $\otimes E_{2p,2q}$, the tensor algebra of $E_{2p,2q}$, and we define the linear map J_1 from $\otimes E_{2p,2q}$ into $\otimes E_{2p,2q}$ by

- $J_1(x_1 \otimes \cdots \otimes x_k) = J(x_1) \otimes \cdots \otimes J(x_k)$,
- $J_1(\lambda) = \lambda$ for all $\lambda \in \mathbf{R}$.

J_1 is well defined. Let $N(Q_{2p,2q})$ be the two-sided ideal generated by the elements $x \otimes x - Q_{2p,2q}(x).1$, where $Q_{2p,2q}$ is the quadratic standard form of signature $(2p, 2q)$ defined on $E_{2pq} =_{\mathbf{R}} \mathbf{H}_{p,q}$:

$$\begin{aligned} &J_1\{x \otimes x - Q_{2p,2q}(x) \cdot 1\} \\ &\quad = J(x) \otimes J(x) - Q_{2p,2q}(x) \cdot 1 \\ &\quad = J(x) \otimes J(x) - Q_{2p,2q}(J(x)) \cdot 1, \qquad \text{since } Q_{2p,2q}(x) = Q_{2p,2q}(J(x)) \end{aligned}$$

(We recall that J is orthogonal for $Q_{2p,2q}$.) J_1 conserves $N(Q_{2p,2q})$, so J_1 induces $\tilde{J}$, a linear map from $C_{2p,2q}$ into itself that has the required properties. We can remark that $C^-_{2p,2q}$ is a **C**-space by setting, for $c \in C^-_{2p,2q}$,

$$\lambda = \alpha + i\beta; \alpha, \beta \in \mathbf{R} : c(\alpha + i\beta) = c\alpha + \tilde{J}(c)\beta$$

($C^-_{2p,2q}$ is equipped with a transfer operator $\tilde{J}$ such that $\tilde{J}^2 = -\mathrm{Id}$ on $C^-_{2p,2q}$.)

We know that to any quadratic automorphism u of $E_{2p,2q}$, there corresponds canonically an automorphism Φ_u of $C_{2p,2q}$. If $u \in SO(2p, 2q)$, then Φ_u is an inner automorphism of $C_{2p,2q}$, and for all $x \in E_{2p,2q}$, $u(x) = \Phi_u(x) = b_u x b_u^{-1}$, where b_u is the product of an even number of regular vectors of $E_{2p,2q}$ and $b_u \in G^+_{2p,2q}$ (the even Clifford group of $C_{2p,2q}$). More precisely, $u = \varphi(b_u)$, where $b_u \in G^+(E_{2p,2q})$ and φ denotes the natural homomorphism from $G(2p, 2q)$ onto $O(2p, 2q)$ associated with the exact sequence (we recall that the Clifford group $G(2p, 2q)$ is the group of all invertible elements of the Clifford algebra such that for any g in $G(2p, 2q)$, for any x in $E_{2p,2q}$, $\varphi(g)x = gxg^{-1} \in E_{2p,2q}$)

$$1 \to \mathbf{R}^* \to G(2p, 2q) \xrightarrow{\varphi} O(2p, 2q) \to 1.$$

Moreover, we notice the following exact sequence:

$$1 \to \mathbf{R}^* \to G^+(2p, q) \xrightarrow{\varphi} SO(2p, 2q) \to 1.$$

3.7.1.2 Theorem *U belongs to $SO(2p, 2q)$ and $Ju = uJ$ if and only if u induces an inner automorphism Φ_u of $C_{2p,2q}$ such that for all $x \in E_{2p,2q}$ there exists $b_u \in G^+_{2p,2q}$ such that $\Phi_u(x) = b_u x b_u^{-1} = u(x)$ and $\tilde{J}(b_u) = b_u$.*

• If $u \in SO(2p, 2q)$ and $Ju = uJ$, then there exists $b_u \in G^+_{2p,2q}$ such that $\Phi_{b_u}(x) = b_u x b_u^{-1} = u(x)$, $b_u = x_1 \cdots x_{2h}$, modulo a scalar in $\mathbf{R}^*$, where the x_i belong to $E_{2p,2q}$, and by definition of J, which is a similarity of $(E_{2p,2q})$ of ratio $\rho = 1$, and of $\tilde{J}$, $\tilde{J}(b_u) = b_u$, since $\rho = 1$.

• Conversely, if u induces an inner automorphism of $C_{2p,2q}$ such that $\Phi_u(x) = b_u x b_u^{-1} = u(x)$, with $b_u \in G^+_{2p,2q}$, then necessarily $u \in SO(2p, q)$. Since $\tilde{J}(b_u) = b_u$, we have $\tilde{J}(b_u^{-1}) = b_u^{-1}$. Then

$$u[\tilde{J}(x)] = b_u \tilde{J}(x) b_u^{-1} = \tilde{J}(b_u x b_u^{-1}) = \tilde{J}(u(x)),$$

and so $uJ = Ju$. We notice that b_u is determined up to a factor in $\mathbf{R}^*$.

3.7.2 Definition of the Clifford Algebra Associated with $\mathbf{H}_{p,q}$

3.7.2.1 Definition We agree to call the Clifford algebra associated with $\mathbf{H}_{p,q}$ the real algebra denoted by

$$Cl^{p,q} = \left\{ g \in C^+_{2p,2q} : \tilde{J}(g) = g \right\} = \left\{ z + \tilde{J}(z), z \in C^+_{2p,2q} \right\}.$$

If we choose the first definition, we notice that for all $g \in C^+_{2p,2q}$, we have that $z = g + \tilde{J}(g)$ is in $Cl^{p,q}$, because $\tilde{J}(z) = z$. Then we remark that if $z \in C^+_{2p,2q}$ and $\tilde{J}(z) = z$, then $z/2 + \tilde{J}(z/2) = z$ and $z/2 \in C^+_{2p,2q}$, whence the result follows. $Cl^{p,q}$ is defined as a subalgebra of $C^+_{2p,2q}$. $\tilde{J}$ is an involutive automorphism of $Cl^{p,q}$.

We recall the following lemma:[16]

3.7.2.2 Lemma *Let E be an n-dimensional vector space over a skew field K and let u be an involution in $GL_n(K)$. If the characteristic of K is not 2, E is a direct sum of two subspaces V and W (one of which may possibly be 0) such that $u(x) = x$ on V and $u(x) = -x$ on W. V and W will be called the plus-subspace and the minus-subspace of u. They determine u completely.*

So $Cl^{p,q}$ appears as the plus-subspace for the automorphism $\tilde{J}$ of $C^+_{2p,2q}$, $\tilde{J}^2 = \mathrm{Id}$ on $C^+_{2p,2q}$. Thus, now $\dim_{\mathbf{R}} Cl^{p,q} \leq \dim_{\mathbf{R}} C^+_{2p,2q} = 2^{2p+2q-1}$.

[16] Cf. J. Dieudonné, On the Automorphisms of the Classical Groups, *Mem. Ann. Math. Soc.*, no. 2, pp. 1–95, 195, page 3, Lemma 2, or *La Géométrie des Groupes*, Springer, 1955, p. 5.

3.7.2.3 Definition Let φ_J be the map from $\mathbf{H}_{p,q}$ into $Cl^{p,q}$ defined by $\varphi_J(x) = x\tilde{J}(x) = xJ(x)$.

• φ_J defines a map from $\mathbf{H}_{p,q}$ into $Cl^{p,q}$ since $xJ(x) \in C^+_{2p,2q}$ and since $\tilde{J}(\varphi_J(x)) = \tilde{J}(x)\tilde{J}^2(x) = J(x)(-x) = x\tilde{J}(x) = \varphi_J(x)$ since $\tilde{J}^2(x) = -x$, and on the other hand, $x\tilde{J}(x) + \tilde{J}(x)x = 2R(x, J(x)) = 0$, since $R(x, \tilde{J}(x) = -I(\tilde{J}(x), \tilde{J}(x)) = 0$, where R denotes the bilinear real symmetric form associated with $Q_{p,q}$ and I the skew-symmetric form defining the symplectic product.

• φ_J is $\mathbf{R}$-quadratic, which means that $\varphi_J(\lambda x) = \lambda^2\varphi_J(x)$ for all $x \in \mathbf{H}_{p,q}$ and for all $\lambda \in \mathbf{R}$, and $\frac{1}{2}\{\varphi_J(x+y) - \varphi_J(x) - \varphi_J(y)\} = \frac{1}{2}\{xJ(y) + yJ(x)\} = \varphi(x, y)$, where φ is an $\mathbf{R}$-bilinear symmetric form from $\mathbf{H}_{pq} \times \mathbf{H}_{p,q}$ into $Cl^{p,q}$. (The verification is easy.) We remark that for all $x \in \mathbf{H}_{p,q}$, $\varphi(x, x) = \varphi_J(x)$, and that for all $x \in \mathbf{H}_{p,q}$, $\varphi_J(Jx) = \varphi_J(x)$. We have the following statement:

3.7.2.4 Theorem *The algebra $Cl^{p,q}$ is the real associative algebra generated by the $\varphi_J(x)$, $x \in \mathbf{H}_{p,q}$, $p \geq 1, q \geq 1$.*

Proof. Let us denote by F the real algebra generated by the $\varphi_J(x)$; for all $x \in \mathbf{H}_{p,q}$ ($\mathbf{H}_{p,q}$ is identified with $(E_{2p,2q}, J)$, F is included into $Cl^{p,q}$. We are going to show that $Cl^{p,q}$ is included in F.

• We notice that for all $x, y \in E$, $\varphi(x, y) \in F$. Then, since for all $x \in \mathbf{H}_{p,q}$, $\varphi(x, Jx) = \frac{1}{2}(-x^2+x^2) = 0, 0 \in F$. Moreover, $(\varphi_J(x))^2 = (xJ(x))^2 = -[Q_{2p,2q}, (x)]^2$. $1 \in \mathbf{R}$. Since $p > 1$ there exists $x_1 \in E_{2p,2q}$ such that $Q_{2p,2q}(x_1) = 1$ and for $z \in \mathbf{R}_+$, $\varphi_J(\sqrt[4]{z}x_1)\varphi_J(\sqrt[4]{z}x_1) \in F$, and on the other hand, $(\varphi_J(\sqrt[4]{z}x_1))^2 = -z$. Thus $-z$ and $z \in F$. (We can also use lemma IV.4 of Deheuvels.[17]) If z is in $\mathbf{R}_-$, $-z = a \in F$, and so $z = -a \in F$. Thus $\mathbf{R} \subset F$.

• We introduce now $C_{2p,2q}(s)$, the space called the space of s-vectors and more precisely $C_{2p,2q}(2s)$ and we want to show by a recurrent method that $Cl^{p,q} \cap C^+_{2p,2q}(2s) \subset F$. $C^+_{2p,2q}(2s)$ is the $\mathbf{R}$-space generated by 1 and by the products $x_1 \cdots x_{2s}$, where $x_i \in E_{2p,2q}$, for all i, $1 \leq i \leq 2s$.

• **Case $s = 1$**

Since $\mathbf{R}$ is included in F, it is enough to show that for all $x, y \in E_{2p,2q}$, $x\tilde{J}(y) + \tilde{J}(x\tilde{J}(y)) \in F$. Indeed, $z \in Cl^{p,q} \cap C^+_{2p,2q}(2s)$ iff $z = x_1x_2 + \tilde{J}(x_1x_2)$, $x_1, x_2 \in E_{2p,2q}$. Since $\tilde{J}^2 = -\mathrm{Id}$ on $E_{2p,2q}$, there exists $y_2 = \tilde{J}(-x_2) = -\tilde{J}(x_2)$ such that $\tilde{J}(y_2) = x_2$. So z is of the form $x\tilde{J}(y) + \tilde{J}(x\tilde{J}(y))$. Moreover, $x\tilde{J}(y) + \tilde{J}(x\tilde{J}(y)) = x\tilde{J}(y) + \tilde{J}(x)\tilde{J}^2(y) = x\tilde{J}(y) - \tilde{J}(x)y$. Since $2R(y, \tilde{J}(x)) = y\tilde{J}(x) + \tilde{J}(x)y$ and then $-\tilde{J}(x)y = -2R(y, \tilde{J}(x) + y\tilde{J}(x))$, it follows that

$$x\tilde{J}(y) - \tilde{J}(x)y = -2R(y, \tilde{J}(x)) + x\tilde{J}(y) + y\tilde{J}(x) = 2\varphi(x, y) - 2R(y, \tilde{J}(x))$$

with $2\varphi(x, y) \in F$ and $2R(y, \tilde{J}(x)) \in F$.

[17] R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., lemma IV.4, p. 139.

• **Case $s > 2$**

Let z be in $C^+_{2p,2q}(2s)$. Let us write $z = u \cdot v$, with $u \in C^+_{2p,2q}(2k)$ and $v \in C^+_{2p,2q}(2l)$ with $k, l < s$. By hypothesis, we can assume that $C^+_{2p,2q}(2t) \cap Cl^{p,q} \subset F$ for all $t < s$. Let us write now

$$uv + \tilde{J}(uv) = \left\{\frac{u}{2} + \tilde{J}\left(\frac{u}{2}\right)\right\}\{v\tilde{J}(v)\} + \{u - \tilde{J}(u)\}\left\{\frac{v}{2} - \tilde{J}\left(\frac{v}{2}\right)\right\} = w_1w_2 + z_1z_2.$$

We verify easily that $\tilde{J}(w_1) = w_1$, $\tilde{J}(w_2) = w_2$ and that $w_1w_2 \in Cl^{p,q}$. We notice that $\tilde{J}(z_1) = -z_1$, $\tilde{J}(z_2) = -z_2$ and then $\tilde{J}(z_1z_2) = z_1z_2$. So $z_1z_2 \in Cl^{p,q}$. According to the recurrence hypothesis, w_1w_2 and z_1z_2 belong to F and $uv + \tilde{J}(uv) \in F$. We have found the formula

$$\text{for all } x, y \in \mathbf{H}_{p,q}, xJ(y) - J(x)y = 2\varphi(x, y) - 2R(J(x), y).$$

3.7.2.5 Lemma *An element $e_{i_1} \cdots e_{i_{2p}}$ of the basis of $C^+_{2p,2q}$ belongs to $Cl^{p,q}$ if and only if $e_{i_1} \cdots e_{i_{2p}}$ is of the form $e_{i_1}J(e_{i_1})e_{i_2}J(e_{i_2}) \cdots e_{i_p}J(e_{i_p})$.*

The proof is straightforward by recurrence and left to the reader.

We can easily verify that $\tilde{J}(1) = 1$ and $\tilde{J}\{e_1J(e_1) \cdots e_nJ(e_n)\} = e_1J(e_1) \cdots e_nJ(e_n)$ and thus 1 and $e_1J(e_1)e_2J(e_2) \cdots e_nJ(e_n)$ belong to $Cl^{p,q}$. If $n = 2$, among the eight elements

$$1, e_1e_2, e_1J(e_1), e_1J(e_2), J(e_1)e_1, J(e_1)J(e_2), e_2J(e_1), e_1J(e_1)e_2J(e_2)$$

of the basis of $C^+_{2p,2q}$, only the following ones belong to $Cl^{1,1}$:

$$1, e_1J(e_1), e_2J(e_1), e_1J(e_1)e_2J(e_2).$$

More precisely, denoting by C_n^j the classical coefficients, there are $1 = C_2^0$ 0-vector, $1 = C_2^2$ 4-vectors, and $2 = C_2^1$ 2-vectors so that the cardinality of the set of basis elements of $Cl^{1,1}$ is now $1 + C_2^1 + 1 = 2^2$. In the same way, more generally, in the case of $Cl^{p,q}$ among the elements of the basis of $C^+_{2p,2q}$, there remains in the set of the basis elements of $Cl^{p,q}$ only $1 = C_n^0$ 0-vector, 1; $1 = C_n^n$ n-vectors; $e_1J(e_1) \cdots e_nJ(e_n)$; $n = C_n^1$ 2-vectors; C_n^2 4-vectors; and C_n^p $2p$-vectors such that

$$\dim_{\mathbf{R}} Cl^{p,q} = 1 + C_n^1 + \cdots + C_n^p + \cdots + C_n^n = 2^n.$$

Thus, we have obtained the following theorem:

3.7.2.6 Theorem *The real associative algebra $Cl^{p,q}$ is of dimension 2^n.*

The study of the periodicity of such an algebra is given as an exercise. See below. The proof of 3.7.2.4 naturally leads us to the following definition.

3.7.3 Definition 2 of the Clifford Algebra Associated with $H_{p,q}$

Let A be an $\mathbf{R}$-associative finite-dimensional algebra with a unit element, and let $\mathbf{H}_{p,q}$ denote $(E_{2p,2q}, J)$.

3.7.3.1 Definition of a Pseudounitary Clifford Mapping

We define a pseudounitary Clifford mapping from $\mathbf{H}_{p,q}$ into A to be any mapping Ψ from $\mathbf{H}_{p,q}$ into A such that

(a) $\Psi(\lambda x) = \lambda^2 \Psi(x)$, for all $\lambda \in \mathbf{R}$,
(b) $(1/2)\{\Psi(x+y) - \Psi(x) - \Psi(y)\} = \varphi(x, y)$, where φ is an $\mathbf{R}$-bilinear mapping from $\mathbf{H}_{p,q} \times \mathbf{H}_{p,q}$ into A,
(c) $(\Psi(x))^2 = -[Q_{2p,2q}(x)]^2 1_A$, for all $x \in \mathbf{H}_{p,q}$.
(d) $\Psi(Jx) = \Psi(x)$ for all $x \in \mathbf{H}_{p,q}$.

We notice immediately that $\Psi(x) = \varphi(x, x)$. We also notice that if B is another associative $\mathbf{R}$ algebra with a unit element and if Φ is a homomorphism of algebras with unit elements from A into B, which means that Φ is $\mathbf{R}$-linear, multiplicative—$(\Phi(aa') = \Phi(a)\Phi(a')$—and that $\Phi(1_A) = 1_B$, then $\Psi_1 = \Phi \circ \Psi$ from $\mathbf{H}_{p,q}$ into B is a pseudounitary Clifford mapping from $\mathbf{H}_{p,q}$ into B. We can easily verify that for all $\lambda \in \mathbf{R}$,

$$\Psi_1(\lambda x) = \lambda^2 \Psi_1(x)$$

and that

$$\frac{1}{2}\{\Psi_1(x+y) - \Psi_1(x) - \Psi_1(y)\} = \varphi_1(x, y),$$

where φ_1 is an $\mathbf{R}$-bilinear from $\mathbf{H}_{p,q} \times \mathbf{H}_{p,q}$ into B, and that for all $x \in \mathbf{H}_{p,q}$,

$$\begin{aligned}(\Psi_1(x))^2 &= (\Phi \circ \Psi(x))^2 = \Phi((\Psi(x))^2) = \Phi(-(Q_{2p,2q}(x))^2 1_A) \\ &= -Q_{2p,2q}(x))^2 1_B.\end{aligned}$$

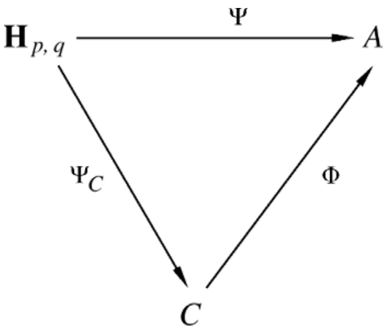
Moreover, $\Psi_1(J_x) = \Phi \circ \Psi \circ J(x) = \Phi \circ \Psi(x) = \Psi_1(x)$, for all $x \in \mathbf{H}_{p,q}$.

3.7.3.2 Definition 2 of the Clifford Algebra Associated with $\mathbf{H}_{p,q}$

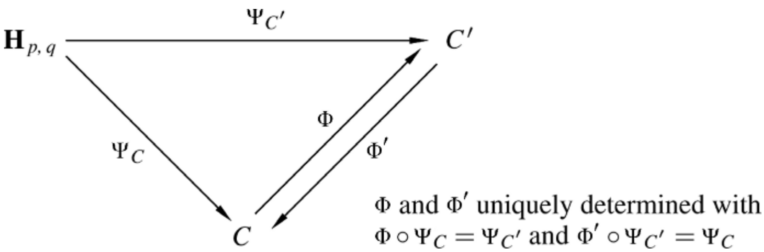
We define a Clifford algebra associated with $\mathbf{H}_{p,q}$ to be any $\mathbf{R}$ associative algebra, with unit element 1_C, equipped with a pseudounitary Clifford mapping Ψ_c from $\mathbf{H}_{p,q}$ into C, which satisfies the following conditions:

(1) $\Psi_c(\mathbf{H}_{p,q})$ generates C,
(2) For any Clifford pseudounitary mapping Ψ from $\mathbf{H}_{p,q}$ into A ($\mathbf{R}$-associative algebra with a unit element), there exists a homomorphism of algebras with

unit elements Φ from C into A such that $\Psi = \Phi \circ \Psi_C$, that is, $(\forall x)$ $(x \in \mathbf{H}_{p,q})$, $\Phi(\Psi_C(x)) = \Psi(x)$.



The second condition expresses that any pseudounitary Clifford mapping of $\mathbf{H}_{p,q}$ can be obtained from the map Ψ_c, which is universal. Consequently, if a pseudo-hermitian standard space $\mathbf{H}_{p,q}$ possesses a Clifford algebra C, it is unique up to isomorphism. Indeed, let C' be another Clifford algebra of $\mathbf{H}_{p,q}$. The diagram



implies that

$$\Phi' \circ \Phi \circ \Psi_C = \Phi' \circ \Psi_{C'} = \Psi_C$$

and

$$\Phi \circ \Phi' \circ \Psi_{C'} = \Phi \circ \Psi_C = \Psi_{C'}.$$

Since $\Psi_C(\mathbf{H}_{p,q})$ generates C and $\Psi_{C'}(\mathbf{H}_{p,q})$ generates C', we can deduce that $\Phi' \circ \Phi = \mathrm{Id}_C$ and$\Phi \circ \Phi' = \mathrm{Id}_{C'}$ Φ and Φ' are isomorphisms that are uniquely determined, each of them the inverse of the other through interchanging Ψ_C in $\Psi_{C'}$ or $\Psi_{C'}$ in Ψ_C. *We can speak of the Clifford algebra of the pseudounitary space* $\mathbf{H}_{p,q}$.

3.7.3.3 Equivalence of Definition 1 and Definition 2

The algebra $Cl^{p,q}$ defined in 3.7.2.1 is such that $(Cl^{p,q}, \varphi_J)$, where $\varphi_J(x) = xJ(x)$ satisfies the conditions given in 3.7.2.2. φ_J is a pseudounitary Clifford mapping according to the results given in the proof of Theorem 3.7.2.4, and satisfies the conditions a,b,c,d of a pseudounitary Clifford mapping.

The converse will be studied below in the exercises. The converse can be studied by using the Lemma given in 3.7.2.5.

Subsequently, to any regular standard pseudo-hermitian space we can associate a Clifford algebra, namely $(Cl^{p,q}, \varphi_J)$.

3.7.4 Clifford Groups and Covering Groups of $U(p, q)$

With notation of Deheuvels,[18] we introduce the covering groups $RO(2p, 2q)$ and $RO^+(2p, q)$ respectively of $O(2p, 2q)$ and $SO(2p, 2q)$, associated with the exact sequences

$$1 \to Z_2 \to RO(2p, 2q) \to O(2p, 2q) \to 1$$

and

$$1 \to Z_2 \to RO^+(2p, 2q) \to SO(2p, 2q) \to 1.$$

We introduce $\mathrm{Spin}\,(2p, 2q) = RO^{++}(2p, q)$, the connected component of the identity in $RO(2p, 2q)$, which is a twofold covering group of $SO^+(2p, q) = O^{++}(2p, 2q)$ associated with the exact sequence

$$1 \to Z_2 \to \mathrm{Spin}\,(2p, 2q) \to SO^+(2p, q) \to 1.$$

For $p > 1, q > 1$ $RO(2p, 2q)$ has four connected components by arcs that are twofold coverings for the corresponding components in $O(2p, q)$.

Let $\tilde{G}_{2p,2q}$ be the regular Clifford group consisting of invertible elements g of the Clifford algebra $C_{2p,2q}$ such that for any x in $E_{2p,2q}$, $\Psi(g)x = \pi(g)xg^{-1} = y \in E_{2p,2q}$ (π is the principal automorphism of $C_{2p,2q}$). Such a group is also the group consisting of products of nonisotropic elements in $E_{2p,2q}$, in the Clifford algebra $C_{2p,2q}$. $\tilde{G}^+_{2p,2q}$ denotes the even regular Clifford group $\tilde{G}^+_{2p,2q} = \tilde{G}_{2p,2q} \cap C^+_{2p,2q}$. We remark that $\varphi = \Psi$ on $C^+_{2p,2q}$, where φ is defined, as usual, by $\varphi(g) \cdot x = gxg^{-1}$.

3.7.4.1 Theorem *For any $v \in U(p, q)$ there exists an invertible element $b_v \in Cl^{p,q}$ determined up to a scalar in R^* such that $\Phi_v(x) = b_v x b_v^{-1} = v(x)$, for all $x \in H_{p,q}$. Conversely, for any invertible b belonging to $Cl^{p,q}$ such that for all $x \in \mathbf{H}_{p,q}$: $bxb^{-1} = y \in \mathbf{H}_{p,q}$, the mapping $x \to bxb^{-1}$ induces an element of $U(p, q)$.*

• The first part is a consequence of Theorem 3.7.1.2 and Definition 3.7.2.1 of $Cl^{p,q}$.

• Conversely, with any invertible element b of $Cl^{p,q}$ such that for all $x \in \mathbf{H}_{p,q}$, $bxb^{-1} = y \in \mathbf{H}_{p,q}$, we can associate $v \in SO(2p, q)$ such that $\Phi_v(x) = b_v x b_v^{-1} = v(x)$, for all $x \in \mathbf{H}_{p,q}$. We introduce J and $\tilde{J}$ defined as before. Then for all $x \in \mathbf{H}_{p,q}$ and for all $y \in \mathbf{H}_{p,q}$, $H_{p,q}$ identified with $(E_{2p,2q}, J)$ such that $Q_{2p,2q}(y) = Q_{2p,2q}(J(y)) \neq 0$ as $bxb^{-1} \in E_{2p,2q}$, we can write

$$\tilde{J}(bxb^{-1}y) = \tilde{J}(bxb^{-1})\tilde{J}(y) = b\tilde{J}(x)b^{-1}\tilde{J}(y),$$

[18] R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit.

i.e.,

$$[\tilde{J}(bxb^{-1}) - b\tilde{J}(x)b^{-1}]J(y) = 0,$$

and according to the hypothesis made for y, we can deduce that for all $x \in E_{2p,2q}$: $\tilde{J}(bxb^{-1}) = b\tilde{J}(x)b^{-1}$, i.e., $J \circ v = v \circ J$ and thus $v \in U(p,q)$ by definition, Ψ is a natural homomorphism from $\tilde{G}_{2p,2q}$ onto $O(2p, 2q)$. The restriction of Ψ to $\tilde{G}^{+}_{2p,2q}$ onto $SO(2p, 2q)$ leads us to a surjective homomorphism with kernel R^* associated with the following exact sequence:

$$1 \to R^* \to \tilde{G}^{+}_{2p,2q} \cap Cl^{p,q} \to U(p,q) \to 1.$$

3.7.4.2 Definition of the Pseudounitary Clifford Group and of the Covering Group $RU(p,q)$ of $U(p,q)$

Definitions $\tilde{G}^{+}_{2p,2q} \cap Cl^{p,q}$, determined up to an isomorphism, is called the pseudounitary Clifford group. $RU(p,q) = RO^{+}(2p,q) \cap Cl^{p,q}$ is called the covering group for $U(p,q)$ associated with the exact sequence

$$1 \to Z_2 \to RU(p,q) \to U(p,q) \to 1.$$

3.7.4.3 Definition of the Spinor Group Spin $U_{p,q}$

We recall the exact following sequence of groups:

$$1 \to Z_2 \to \text{Spin}\,(2p, 2q) \to SO^{+}(2p, 2q) \to 1.$$

3.7.4.4 Definition $\text{Spin}(2p, 2q) \cap Cl^{p,q}$ is called, by definition, the spinor group associated with $H_{p,q}$ and is denoted by Spin $U(p,q)$. We define $\Psi(\text{Spin}\, U_{p,q}) = U_0(p,q)$ as the reduced pseudounitary group. We have the following exact sequence:

$$1 \to Z_2 \to \text{Spin}\, U_{p,q} \to U_0(p,q) \to 1.$$

3.7.5 Fundamental Diagram Associated with $RU(p,q)$

3.7.5.1 General Definitions

Following a method initiated by Atiyah, Bott, and Shapiro,[19] we introduce the following definition:

3.7.5.1.1 Definition Let $A(Q)$ be one of the classical groups $RO(Q)$ (covering group of $O(Q)$); $G(Q)$ (Clifford group); Spin Q. Set $A^u(Q) = A(Q) \times_{Z_2} U(1)$, where Z_2 acts on $A(Q)$ and $U(1)$ as $\{\pm 1\}$.

[19] M. F. Atiyah, R. Bott, and A. Shapiro, Clifford modules, *Topology*, vol. 3 Suppl. 1, pp. 3–38, 1964.

We recall that $U(1)$ is the classical group of complex numbers z with $|z| = 1$ (for the multiplicative law). We recall the following definition and theorem:

3.7.5.1.2 Definition and Theorem

Let $(E^c Q^c)$ be the complexification of (E, Q), where (E, Q) is a standard regular quadratic space such that $(Cl(Q))^c \simeq Cl(E_c, Q^c)$. Let $G^c(E, Q)$ be the subgroup of invertible elements g of $(Cl(Q))^c$ that satisfy

$$\forall y \in E, \quad \pi(g) y g^{-1} \in E,$$

and let $RO^c(E, Q)$ be the kernel of the spinor norm. We have the following exact sequence:[20]

$$1 \to U(1) \to RO^c(E, Q) \to^{\delta} O(Q) \to 1.$$

3.7.5.1.3 Corollary *We have a natural isomorphism*

$$RO^c(E, Q) \simeq RO(E, Q) \times_{Z_2} U(1).$$

We recall the following exact sequences:

$$1 \to Z_2 \to RO(2p, 2q) \to O(2p, 2q) \to 1,$$

$$1 \to Z_2 \to RO^+(2p, 2q) \to SO(2p, 2q) \to 1,$$

$$1 \to Z_2 \to RU(p, q) \xrightarrow{\varphi = \Psi} U(p, q) \to 1,$$

3.7.5.1.4 Definitions

Let us introduce

$$\alpha : z \to \alpha(z) = z^2 \text{ from } U(1) \text{ into } U(1)$$

with

$$\alpha' : [v, u] \in RU(p, q) \times_{Z_2} U(1) \to \alpha'([v, u]) = u^2 \in U(1),$$

where $[v, u]$ denotes the class of $(v, u) \in RU(p, q) \times_{Z_2} U(1)$, $\delta : \delta[g, z] = \Psi(g)$, and $i : i(g) = [g, 1]$ for all $g \in RU(p, q)$ and all $z \in U(1)$.

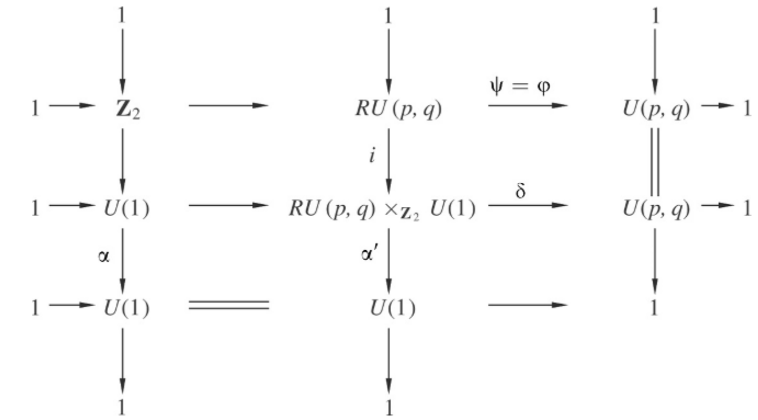
We have the following statement:

3.7.5.1.5 Proposition *We have the following commutative diagram of Lie groups associated with $RU(p, q)$:*

[20] In fact, in their paper, M. F. Atiyah, R. Bott, and A. Shapiro found the following exact sequence:

$$1 \to U(1) \to \mathrm{Pin}^c(k) \to O(k) \to 1,$$

where $O(k)$ is the orthogonal group of $\mathbf{R}^k$ provided with a negative definite quadratic form and where $U(1)$ is the subgroup consisting of elements $1 \otimes z \in C_k \otimes_{\mathbf{R}} \mathbf{C}$ with $|z| = 1$, and where C_k is the corresponding Clifford algebra of $\mathbf{R}^k$. We identify here such a subgroup with $U(1)$, which is classical.



3.7.6 Characterization of $U(p, q)$

3.7.6.1 Let Us Assume That $p + q = n = 2r$, $p \leq n - p$ (cf. below 3.13 Appendix)

We consider the basis $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ of $_{\mathbf{R}}(C^n)$ and we introduce, as previously, $Q_{2p,2q}$ (of signature $(2p, 2q)$), the quadratic form associated with the bilinear real symmetric form R. Let $E_{2n} = \mathbf{R}^{2n}$ and let E'_{2n} be the complexification of $\mathbf{E}_{2n}$, a $2n$-dimensional C-space $(2n = r)$.

We know that there exists a special Witt decomposition of E'_{2n}, $E'_{2n} = F + F'$, where F, respectively F', is a maximal totally isotropic $2r$-dimensional subspace. We write $F = \{x_1, \ldots, x_n\}$, $F' = \{y_1, \ldots, y_x\}$ with respective explicit bases

$$x_1 = \frac{e_1 + e_n}{2}, \ldots, x_p = \frac{e_p + e_{n-p+1}}{2}, x_{p+1} = \frac{ie_{p+1} + e_{n-p}}{2}, \ldots,$$

$$x_r = \frac{ie_r + e_{n-r+1}}{2}, x_{r+1} = \frac{J(e_1) + iJ(e_n)}{2}, \ldots,$$

$$x_{r+p} = \frac{J(e_1) + J(e_{n-p+1})}{2}, x_{r+p+1} = \frac{iJ(e_{p+1}) + J(e_{n-p})}{2}, \ldots,$$

$$x_n = \frac{iJ(e_r) + J(e_{n-r+1})}{2},$$

$$y_1 = \frac{e_1 - e_n}{2}, \ldots, y_p = \frac{e_p - e_{n-p+1}}{2}, y_{p+1} = \frac{ie_{p+1} - e_{n-p}}{2}, \ldots,$$

$$y_r = \frac{ie_r - e_{n-r+1}}{2}, y_{r+1} = \frac{J(e_1) - J(e_n)}{2}, \ldots,$$

$$y_{r+p} = \frac{J(e_p) - J(e_{n-p+1})}{2}, \dots, y_{r+p+1} = \frac{iJ(e_{p+1}) - J(e_{n-p})}{2}, \dots,$$

$$y_n = \frac{iJ(e_n) - J(e_{n-r+1})}{2},$$

with for $1 \leq j \leq p$, $\bar{x}_j = x_j$, $\bar{y}_j = y_j$, for $r+1 \leq j \leq r+p$, $\bar{x}_j = x_j$, $\bar{y}_j = y_j$, for $p+1 \leq j \leq r$, $\bar{y}_j = -x_j$, and for $r+p+1 \leq j \leq n$, $\bar{y}_j = \bar{x}_j$, and with $B(x_i, y_j) = \delta_{1\gamma}/2$, $B(x_i, x_j) = B(y_i, y_j) = 0$ and thus $x_i y_j + y_j x_i = \delta_{ij}$, $1 \leq i, j \leq n$. We recall that $J|F = i\,\mathrm{Id}$ and that $J|F' = -i\,\mathrm{Id}$.

3.7.6.2 Characterization of $U(p,q)$ $(p+q=n=2r,\ p \leq q,\ p \leq r)$

Let us consider $C'_{2p,2q}$ the complexification of $C_{2p,2q}$. As usual, we define $\exp(\lambda X)$, $\lambda \in C$, for $X \in Cl'(2p,2q)$. We know that if $XY = YX$, then $\exp X \exp Y = \exp(X+Y)$, $(\exp X)^{-1} = \exp(-X)$, where $\exp X = \sum_{k \geq 0}(X^k/k!)$, and that $\tau(\exp X) = \exp(\tau(X))$, where τ denotes the principal antiautomorphism of the Clifford algebra. We recall that $U(p,q)$ is the set of elements u of $SO(2p,q)$ such that $u \circ J = J \circ u$. We want to prove the following statement. Let $i \in C : i^2 = -1$, and let $t \in \mathbf{R}$. We denote by Ψ the classical projection already considered: $\Psi(g)x = \pi(g)xg^{-1}$.

3.7.6.2.1 Proposition *$\Psi \exp[it \sum_{k=1}^n (x_k y_k)]$ induces the mappings $x \to e^{it}x$ on F and $x \to e^{-it}x$ on F'.*

Proof.

3.7.6.2.1.1 Lemma

$$(x_k y_k)(x_l y_l) = (x_l y_l)(x_k y_k)$$

and

$$\exp(it\, x_k y_k)\exp(it\, x_l y_l) = \exp(it(x_k y_k x_l y_l)).$$

The result is quite straightforward.

3.7.6.2.1.2 Lemma *Let*

$$z = \exp\left(it\left[\sum_{k=1}^{n}(x_k y_k)\right]\right) = \prod_{k=1}^{n}\exp[it\, x_k y_k].$$

Then $N(z) = e^{int} = (e^{it})^n$ and $|N(z)| = 1$.

Let us consider now the plane generated by x_1 and y_1 such that $x_1^2 = 0$, $y^2 = 0$, and $2R(x,y) = 1$. It is easy to verify that $\tau(\exp it\, x_1 y_1) = \tau(z_1) = \exp(it\, y_1 x_1)$ and that $N(z) = e^{it}$. Thus $|N(z)| =$ and $z_1^{-1} = e^{-it}\exp(it\, y_1 x_1)$. Then it is easy to verify that $e^{-it}\exp(it\, y_1 x_1)x_1\exp(it\, y_1 x_1) = e^{it}x_1$ and that $e^{-it}\exp(it\, x_1 y_1)y_1 \cdot \exp(it\, y_1 x_1) = e^{-it}y_1$. The result is quite obvious by recurrence.

3.7.6.2.1.3 Corollary

$$\Psi \circ \exp\left\{ i\pi \left(\sum_{k=1}^{n} x_k y_k \right) \right\} = -\mathrm{Id} \textit{ on } (\mathbf{R}^{2n})',$$

$$\Psi \circ \exp\left\{ 2i\pi \left(\sum_{k=1}^{n} x_k y_k \right) \right\} = \mathrm{Id} \textit{ on } (\mathbf{R}^{2n})',$$

$$\Psi \circ \exp\left\{ i\frac{\pi}{2} \left(\sum_{k=1}^{n} x_k y_k \right) \right\} = J.$$

3.7.6.2.2 Proposition *The group $U(p,q)$ is identical to $\Psi(\triangle U_{p,q})$, where $\triangle U_{p,q}$ is the set of products of elements z with $|N(z)| = 1$ such that $z = \exp(i\lambda)\cdot \exp(ia^{kl} x_k y_l)$ (with summation in k and l), $a^{kl} \in \mathbf{C}$, with $a^{kl} = \bar{a}^{lk}$ and $\lambda = -(\sum a^{kk})/2$.*

Proof.

• First, it is easy to verify that $U(p,q)$ is included in $\Psi(\triangle U_{p,q})$. It is enough to notice that $x_k y_k$ commutes with $\sum_l x_l y_l$ and to use the previous corollary to express a condition of reality, using the fact that for $z = \exp(i\lambda)\exp(ia^{kl} x_k y_l) = \mu \exp(u)$, with $\mu = \exp(i\lambda)$ and $u = ia^{kl} x_k y_l$; $N(z) = \mu^2 \exp(u + \tau(u)) = \mu^2 \exp(ia^{kl}(x_k y_l + y_l x_k)) = \mu^2 \exp[ia^{kl}\delta_{kl}] = \exp(2i\lambda)\exp(i\sum_k a^{kk})$.

• Then we notice that $U(p,q)$ is connected. Let us consider $\Psi \circ \exp(it\, x_k y_k)$. The result is obtained by considering the value of the norm and the fact that the exponential map generates the connected component of the identify of a Lie group.

3.7.6.2.3 Remark Previously, we assumed that $n = p + q = r$. If $n = 2r + 1$, then $2n = 2r + 2$ is even and we can consider a special Witt decomposition of $E'_{2n}(2p, 2q)$ that leads to the same conclusions. We notice the following exact sequence:

$$1 \to U(1) \to \triangle U(p,q) \to U(p,q) \to 1.$$

So $\triangle U(p,q)$ is isomorphic to $RU(p,q) \times_{Z_2} U(1)$, which gives an algebraic characterization of $RU(p,q) \times_{Z_2} U(1)$.

3.7.7 Associated Spinors

First, we recall the following classical results.[21]

3.7.7.1 A Recall

Let (E, q) be a quadratic regular complex space. If dim $E = 2k$, the Clifford algebra $C(E, Q)$ is isomorphic to $m(2^k, \mathbf{C})$. If dim $E = 2k+1$, the Clifford algebra $C(E, Q)$ is isomorphic to $m(2^k, \mathbf{C}) \oplus m(2^k, \mathbf{C})$.

[21] R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit., p. 331.

3.7.7.2 Another Recall

We have introduced $E_{2n} = R^{2n}$ endowed with a quadratic form of signature $(2p, 2q)$ and the complexification E'_{2n}, a $2n$-dimensional complex space with its own Clifford algebra isomorphic to the complexification of the Clifford algebra of E_{2n}.

Inside this Clifford algebra $[C(R^{2n}, Q_{2p,2q})]'$ we have considered the group $\triangle U(p,q) \simeq RU(p,q) \times_{Z_2} U(1)$, associated with the exact sequence

$$1 \to U(1) \to \triangle U(p,q) \to U(p,q) \to 1.$$

According to the previous result, since $\dim_{\mathbf{C}} E'_{2n} = 2n$, the Clifford associated algebra A is isomorphic to $m(2^n, \mathbf{C})$. A is identical to $L_C(S)$, where S is a complex $2^n =$ dimensional space, a minimal module[22] of such an algebra A. A is a central simple complex algebra.

3.7.7.3 Definition S is by definition the space of spinors associated with such an algebra: $\dim_C S = 2^n$.

3.7.7.4 Pseudo-Hermitian Structure on *S*

A. Weil has shown[23] that for an antilinear involution α over A, a central simple complex algebra, if we denote by $l(a)$ the endomorphism $x \to ax$ of the underlying vector space to A and if we consider the trace $\mathrm{Tr}\, l(a)$, the form $(x, y) \in A \to \mathrm{Tr}\, l(x^\alpha y)$ is a nondegenerate hermitian form associated with the antilinear involution α. R. Deheuvels has shown[24] that α determines on S a pseudo-hermitian scalar product for which α is precisely the operator of adjunction. Moreover, Deheuvels proved that the signature of the corresponding quadratic hermitian form (associated with $(x, y) \to \mathrm{Tr}\, l(x^\alpha y)$) is $(r^2 + s^2, 2rs)$. Let us choose now for $\alpha : \tau$ the principal antiautomorphism of the Clifford algebra A, a central simple complex algebra for which τ is antilinear. Let us take again the proof given by Deheuvels.[24] It is easy to see that the pseudo-hermitian form is a neutral one, $r^2 + s^2 = 2rs$, i.e., $r = s$. So the pseudo-hermitian scalar product on S is neutral of signature $(2^{n-1}, 2^{n-1})$.

The pseudounitary group of automorphisms of S that conserve such a scalar product consists of elements u of $L_C(S) \simeq A \simeq m(2^n, \mathbf{C})$ such that $u^\tau u = 1$. After embedding $RU(p,q)$ into the complexified algebra A by the canonical injection, we obtain that $RU(p,q)$ is contained in $U(2^{n-1}, 2^{n-1})$. We want to show that for $p \geq 2$, Spin $U(p,q)$ is in fact contained in $SU(2^{n-1}, 2^{n-1})$.

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[22] C. Chevalley, (a) *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954; (b) The construction and study of certain important algebras, *Math. Soc. Japan*, 1955. R. Deheuvels, *Formes Quadratiques et Groupes Classiques*, op. cit.

[23] A. Weil, Algebras with involutions and the classical groups, *Collected Papers, vol. II*, (1951–1964), p. 413–447, reprinted by permission of the editors of *J. Ind. Math. Soc.*, Springer Verlag, New York, 1980.

[24] R. Deheuvels, *Groupes Conformes et Algèbres de Clifford*, op. cit.

Proof. Any element $g \in \mathrm{Spin}\,U(p,q)$ is the product of an even number of vectors u_i such that $N(u_i) = 1$ and of an even number of vectors u_j such that $N(u_j) = -1$, $g = u_1 u_2 \cdots u_{2k}$. Since $u_1 u_2 = u_2(u_2^{-1} u_1 u_2)$ and since $y_1 = u_2^{-1}\, u_1 u_2 \in E_{2n}$ with $N(y_1) = N(u_1)$, we can assume that the u_i with $N(u_i) = -1$, if they exist, are set before in the writing of g. Moreover, if two "u_i" are linearly dependent, using previous permutations, we are led to a factor ± 1. So, we can assume that $g = u_1 \cdots u_{2k}$ with u_i linearly independent, two by two, with $N(u_i) = -1$ before, if they exist.

If (u_i) satisfy $(u_i^2) = 1 = N(u_i)$, u_i is an involutive operator of S, and so its determinant equals ± 1. Let us consider two consecutive vectors u_1, u_2, linearly independent, with $N(u_1) = N(u_2) = -1$, and let P be the plane that they generate. If $p \geq 1$ (in fact $2p \geq 2$) there exists $z \in E_{2n}$ such that $R(z,z) = 1$, $R(z,u_1) = R(z,u_2) = 0$, and $(zu_1)^2 = 1, (zu_2)^2 = 1$, $zu_1 zu_2 = -u_1 u_2$. So zu_1, like zu_2, is an involutive operator of S, with determinant equal to ± 1 (cf. Appendix 1.9). Thus, any $g \in \mathrm{Spin}\,U(p,q)$ is the product of elements that have determinant equal to ± 1. So, $\mathrm{Spin}\,U(p,q)$ is contained in the subgroup of the pseudounitary group consisting of elements of determinant ± 1, but since $\mathrm{Spin}\,U(p,q)$ is connected, all these elements have 1 as determinant.

We have obtained the following theorem.

3.7.7.5 Theorem *The space S of spinors associated with A inherits a natural complex structure and a pseudounitary neutral scalar product of signature* $(2^{n-1}, 2^{n-1})$, *up to a scalar factor, which is conserved by the group* $\mathrm{Spin}\,U(p,q)$. *We have the following embedding:* $\mathrm{Spin}\,U(p,q)$ *is contained in* $SU(2^{n-1}, 2^{n-1})$.

3.8 Natural Embeddings of the Projective Quadrics $\tilde{H}_{p,q}$

The embedding can be made as previously (cf. Chapter 1). Let S be the space of spinors previously introduced. Let $[\,|\,]$ be a scalar product on S associated with the involution τ, i.e., for $a \in Cl^{p,q}$, a and a^τ linear operators of S are adjuncts of each other relative to the scalar product $[\,|\,]$. The injective mapping {isotropic line $\{\lambda x\}$ in $H_{p,q}$} $\to$ {maximal totally isotropic subspace $S(x) = \mathrm{Im}(xy)_S = \ker(yx)_S$}, where $(xy)_S$ and $(yx)_S$ are the projectors of S defined by the elements xy and yx of $Cl^{p,q}$, determines a natural embedding of the projective quadric $\tilde{H}_{p,q}$ into the Grassmannian of half-dimensional subspaces $G(S, \frac{1}{2}\dim S)$.

According to general results of Porteous,[25] we obtain that $\tilde{H}_{p,q}$ is homeomorphic to $U(2^{n-1})$. Then we have the following summary statement:

3.8.1 Theorem $\mathrm{Spin}\,U(p,q)$ *is included in* $SU(2^{n-1}, 2^{n-1})$; $\tilde{H}_{p,q}$ *is homeomorphic to* $U(2^{n-1})$.

[25] I. R. Porteous, *Topological Geometry*, 2nd edition, Cambridge University Press, 1981. (Th. 12-19, p. 237; Prop. 17-46 p. 358)

3.9 Covering Groups of the Conformal Pseudounitary Group

3.9.1 A Review of Previous Results

In 3.7.5.2 we found a fundamental diagram, where clearly there appear two exact sequences:

$$1 \to \mathbf{Z}_2 \to RU(p,q) \xrightarrow{\varphi=\Psi} U(p,q) \to 1,$$

$$1 \to U(1) \to RU(p,q)\times_{\mathbf{Z}_2} \xrightarrow{\delta} U(p,q) \to 1.$$

We have previously given in 3.4 a "geometrical construction" of the conformal pseudounitary group defined as $PU(F) = U(p+1,q+1)/U(1)\cdot I$, via the study (cf. 3.3) of an injective mapping from $E_n(2p+2, 2q+2)$ into the isotropic cone $Q(F)$, where $F = \mathbf{H}_{p,q} \oplus \mathbf{H}_{1,1}$ is defined explicitly by

$$\begin{aligned} p_1(X) = {} & \frac{r(x)}{2\sqrt{2}}(e_0 + J(e_0) + e_{n+1} + J(e_{n+1})) + X \\ & - \frac{1}{2\sqrt{2}}(e_0 + J(e_0) - e_{n+1} - J(e_{n+1})), \end{aligned}$$

or equivalently by

$$p_1(X) = \frac{1}{2\sqrt{2}}(r(x)-1)(e_0 + J(e_0)) + x + \frac{1}{2\sqrt{2}}(r(x)+1)(e_{n+1} + J(e_{n+1})).$$

Thus we put

$$p_1 : x \in E \to p_1(x) = r(x)\delta_0' + x - \mu_0',$$

where

$$\begin{aligned} \delta_0' &= \frac{1}{2\sqrt{2}}(e_0 + J(e_0) + e_{n+1} + J(e_{n+1})),\ \mu_0' \\ &= \frac{1}{2\sqrt{2}}(e_0 + J(e_0) - e_{n+1} - J(e_{n+1})), \end{aligned}$$

which constitutes an isotropic base of $\mathbf{H}_{1,1}$ with $2f(\delta_0', \mu_0') = 1$. Then, the geometrical construction of $PU(F)$ was given.

3.9.2 Algebraic Construction of Covering Groups for *PU*(*F*)

3.9.2.1 Preliminary Remark

Let us take again $p_1 : E \to Q(F)$. $p_1(x)$ can have any isotropic direction except that of δ_0', unless we would assume that this direction is obtained as the limit of $r(x)$ tending to infinity.

Let $y = \lambda x + \alpha\delta_0' + \beta\mu_0'$ be an isotropic vector with $\lambda \neq 0$. According to 3.3.1 we have that $\alpha\bar{\beta} + \bar{\alpha}\beta + 2|\lambda|^2 r(x) = 0$. Therefore y has the same direction as $p_1(x)$ if we take $\frac{r(x)}{\alpha} = \frac{1}{\lambda} = \frac{-1}{\beta}$, that is $\alpha = \lambda r(x),\ \beta = -\lambda$.

3.9.2.2 Proposition *There exists a morphism of groups φ_1 with discrete kernel A_1 from $RU(p+1, q+1)$ onto $PU(F) : g \to \varphi_1(g)$ such that for almost all $x \in \mathbf{H}_{p,q}$ and for all $g \in RU(p, q)$,*

$$\pi(g).p_1(x)g^{-1} = \psi(g).p_1 = \sigma_g(x)p_1[\varphi_1(g)x],$$

where $\sigma_g(x)$ is a scalar (A). $\varphi_1(\mathrm{Spin}\, U(p, q))$ is called the real conformal pseudounitary restricted group.

The proof is the same as previously in 2.4.3 and will be studied in the exercises below. The only difficulty is the search and the determination of the kernel.

3.9.2.3 Determination of the Kernel A_1 of φ_1

We will use the following classical lemma.[26] (Cf. exercises, below.)

3.9.2.3.1 Lemma *If a quadratic n-dimensional regular space (E, q) with $n \geq 3$ has isotropic elements and if $\sigma \in O(q)$ fixes every isotropic line, then $\sigma = \pm \mathrm{Id}_E$.*

3.9.2.3.2 Proposition *The kernel of $\varphi_1 : RU(p+1, q+1) \to PU(F)$ is*

$$\{1, -1, E_N, -E_N\} = A_1,$$

where $E_N = e_0 J(e_0) e_1 J(e_1) \cdots e_n J(e_n) e_{n+1} J(e_{n+1})$.

If g belongs to $\mathrm{Ker}\, \varphi_1$, then $\pi(g).p_1(x)g^{-1} = \sigma_g(x)p_1(x)$, for almost all x in $U(p, q)$. Thus, $\varphi_1(g)$ fixes every isotropic line, taking account of the preliminary remark 3.9.2.1. Thus, $\pi(g)zg^{-1} = \pm z$, for any z in $\mathbf{H}_{p,q}$. Therefore $g \in \mathrm{Ker}_{\varphi_1}$ if and only if $\psi(g) = \mathrm{Id}_F$ or $\psi(g) = -\mathrm{Id}_F$.

Since classically $\mathrm{Ker}\, \psi$ is isomorphic to $\mathbf{Z}_2$, if $g \in RU(p+1, q+1)$ satisfies $\psi(g) = -\mathrm{Id}_F$, we know that $\psi(E_N) = -\mathrm{Id}_F$, whence $\psi(gE_N) = \psi(g)\psi(E_N) = \mathrm{Id}_F$, and then $gE_N = \pm 1$, that is, $g = \varepsilon E_N$ with $\varepsilon = \pm 1$. Thus we have obtained the following exact sequence:

$$1 \to A_1 \to RU(p, q) \to CU_n(p, q) \to 1.$$

One can verify that if $(E_N)^2 = 1$, A_1 is isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2$, and if $(E_N)^2 = -1$, A_1 is isomorphic to $\mathbf{Z}_4$. An easy computation gives $E_N^2 = (-1)^n$. Thus if n is even, $A_1 \simeq \mathbf{Z}_2 \times \mathbf{Z}_2$, and if n is odd $A_1 \simeq \mathbf{Z}_4$.

3.9.2.4 Another Covering Group of $PU(F) = CU_n(p, q)$

3.9.2.4.1 Another Fundamental Exact Sequence

As in 3.7.5.1 and 3.7.5.1.4, we can deduce from the above exact sequence the following one:

$$1 \to A_1 \times_{\mathbf{Z}_2} U(1) \to RU(p+1, q+1) \times_{\mathbf{Z}_2} U(1) \to CU_n(p, q) \to 1,$$

[26] E. Artin, *Algèbre Géométrique*, Gauthier-Villars, Paris, 1972, p. 126, Theorème 3.18.

i.e.,

$$1 \to A_1 \times_{\mathbf{Z}_2} U(1) \to \Delta U(p+1, q+1) \to CU_n(p,q) \to 1.$$

3.9.2.4.2 Proposition *We have the following commutative fundamental diagram of Lie groups with the same notation as in 3.7.5.1.4 and 3.7.5.1.5:*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & A_1 & \longrightarrow & RU(p+1,\, q+1) & \longrightarrow & CU_n(p,q) & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & A_1 \times_{\mathbf{Z}_2} U(1) & \to & RU(p+1,\, q+1) \times_{\mathbf{Z}_2} U(1) & \longrightarrow & CU_n(p,q) & \to & 1 \\
 & & \alpha' \downarrow & & \alpha' \downarrow & & \downarrow & & \\
1 & \to & U(1) & = & U(1) & \longrightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

3.9.3 Conformal Flat Geometry ($n = p + q = 2r$)

Let us consider again $E_n(p,q)$, the standard pseudo-Euclidean space, with $p \le q$, $p \le r$. We can introduce the following Witt decompositions of E'_n and $E'_2(1,1)$,[27] respectively (where the $'$ indicates that we consider the complexification of the space):

$$\begin{cases}
x_1 = \frac{e_1+e_n}{2}, \ldots, x_p = \frac{e_p+e_{n-p+1}}{2}, x_{p+1} = \frac{ie_{p+1}+e_{n-p}}{2}, \ldots, \\
x_r = \frac{ie_r+e_{n-r+1}}{2}, x_0 = \frac{e_0+e_{n+1}}{2} \\
y_1 = \frac{e_1-e_n}{2}, \ldots, y_p = \frac{e_p-e_{n-p+1}}{2}, y_{p+1} = \frac{ie_{p+1}-e_{n-p}}{2}, \ldots, \\
y_r = \frac{ie_r-e_{n-r+1}}{2}, y_0 = \frac{e_0-e_{n+1}}{2},
\end{cases}$$

such that for any i and j we have

$$B(x_i, y_j) = \frac{\delta_{ij}}{2} \text{ and } x_i y_j + y_j x_i = 2B(x_i, y_j) = \delta_{ij}, \quad 0 \le i \le r,\ 0 \le j \le r.$$

[27] C. Chevalley, *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954, pp. 13–15, for example, and p. 91, also cf. A. Crumeyrolle, Bilinéarité et géométrie affine attachées aux espaces de spineurs complexes Minkowskiens ou autres, *Annales de l'I.H.P., Section A*, vol. XXXIV, no 3, 1981, p. 351–372.

3.9.3.1 Lemma *Let* $\{x_i, y_j\}_{1\le j\le r}^{1\le i\le r}$ *be a special Witt basis of* E'_n. *Let* $x'_{2k-1} = x_k - y_k$ *and* $x'_{2k} = x_k + y_k$, $k = 1, 2, \ldots, r$. *The* $\{x'_\alpha\}$ *constitute an orthogonal basis of* E'_n *such that if* $z = x'_1 \cdots x'_n$, $z^2 = 1$.[28]

The proof will be recalled below in the exercises.

3.9.3.2 Lemma *Let* $f_r = y_1 \cdots y_r$. *Then* $z f_r = (-1)^r f_r$ *and* $f_r z = f_r$.[29]

The proof is easy and will be given below in the exercises.

3.9.3.3 Lemma *Let* $f_{r+1} = y_1 \cdots y_r y_0$ *be an isotropic* $(r+1)$*-vector of* E_{n+2}. *Then* $e_N f_{r+1} = (-i)^{r-p} f_{r+1}$, *where* $e_N = e_0 e_{n+1} e_1 \cdots e_n$ *and* $f_{r+1} e_N = (-1)^{r+1} \cdot (-i)^{r-p} f_{r+1}$.[30]

The proof will be given below in the exercises.

3.9.3.4 Corollary *Let us consider* $C^+_{2p,2q}$. *We recall that* $Cl^{p,q}$ *is the real algebra defined as*

$$\{g \in C^+_{2p,2q} : \tilde{J}(g) = g\} = \{z + \tilde{J}(z), z \in C^+_{2p,2q}\} \quad \text{(cf. 3.7.2.1 above).}$$

Let us consider an isotropic $(2r+1)$*-vector* $f_{2r+1} = y_1 \cdots y_{2r} y_0$. *Then we have* $E_N f_{2r+1} = (-1)^{r-p} f_{2r+1}$ *and* $f_{2r+1} E_N = (-1)^{r+1-p} f_{2r+1}$.

The proof will be given below in the exercises.

3.9.3.5 Explicit Construction of an Isomorphism from $PU(p+1, q+1)$ onto $CU_n(p, q)$

The construction is made in the same pattern as that given in Chapter 2.[31] We are going to construct explicitly a surjective morphism φ_1 of the Lie group $U(p+1, q+1)$ onto $CU_n(p, q)$ with kernel A_1, where $A_1 = \{1, -1, E_N, -E_N\}$ with $E_N = e_0 J e_0 e_1 J e_1 \cdots e_n J e_n e_{n+1} J e_{n+1}$ such that we have the following diagrams (see Figures 3.1 and 3.2), where i_1 is the identical mapping from $U(p, q)$ into $CU_n(p, q)$, j_1 is the identical mapping from $U(p, q)$ into $U(p+1, q+1)$, X_1 is an isomorphism from $PU(p+1, q+1)$ onto $CU_n(p, q)$, constructed below, $\theta_1 = (X_1)^{-1}$ is an isomorphism from $CU_n(p, q)$ onto $PU(p+1, q+1)$, and η_1 is defined as $\theta_1 \circ \varphi_1$.

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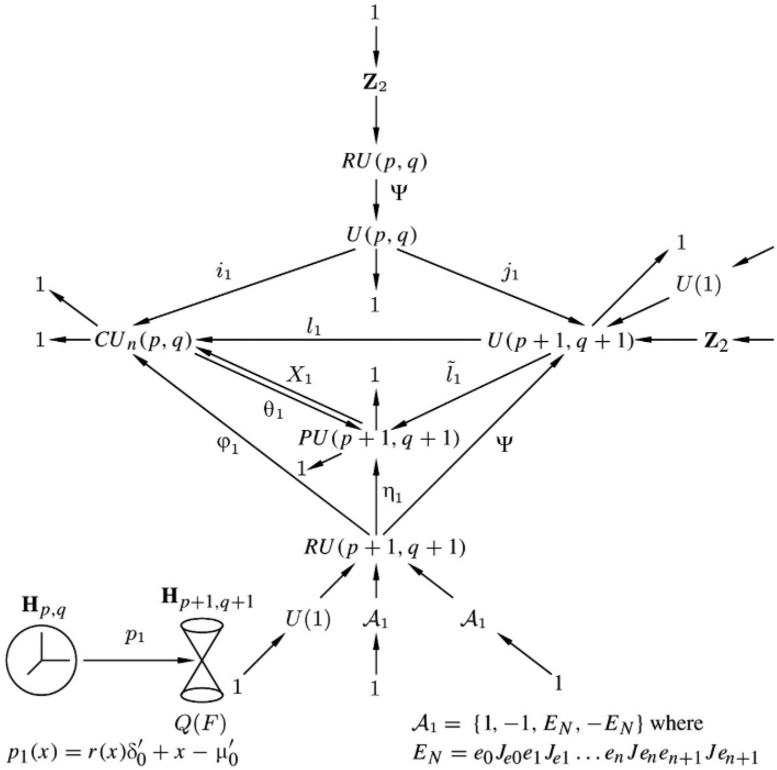
[28] C. Chevalley, *The Algebraic Theory of Spinors*, op. cit., p. 91.

[29] C. Chevalley, *The Algebraic Theory of Spinors*, op. cit., p. 91.

[30] P. Anglès, (a) Les structures spinorielles conformes réelles, Thesis, Université Paul Sabatier, Toulouse, 1983, pp. 40–42; (b) Real conformal spin structures, *Scientiarum Mathematicarum Hungarica*, 1988, pp. 115–139, p. 118.

[31] P. Anglès, (a) *Les Structures Spinorielles Conformes Réelles*, op. cit., pp. 45–50; (b) *Real Conformal Spin Structures*, op. cit., pp. 116–118.



$l_1(w) = f_1$ such that $f_1(x) = \lambda_w(x)\{w.p_1(x) - 2R(p_1(x), \mu_0')\delta_0'\} + \mu_0'$
$2\lambda_g(x) R(w.p_1(x), \mu_0') = -1$
$\lambda_g(x) = (\sigma_g(x))^{-1}$ for $\sigma_g(x) \neq 0$

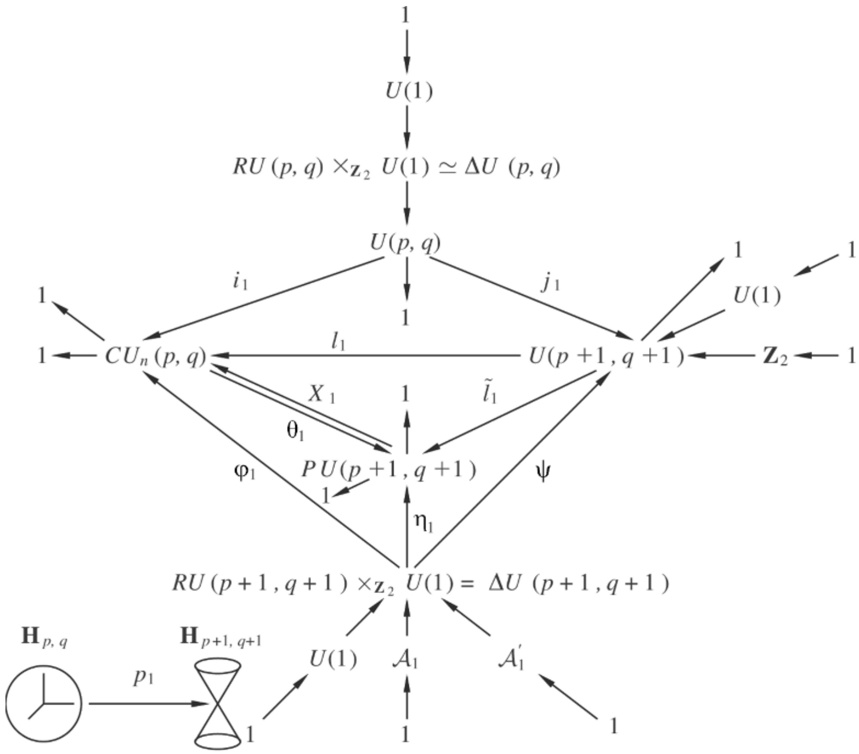
$\mathbf{H}_{p,q} \simeq (\mathbf{R}^{2n}(2_p, x_q), J)$
$p_1(x) = \frac{1}{2\sqrt{2}}(r(x) - 1)(e_0 + J(e_0)) + \frac{1}{2\sqrt{2}}(r(x) + 1)(e_{n+1} + J(e_{n+1})) + x$
$\pi(g) p_1(x) g^{-1} = \sigma_g(x) p_1(f_1(x)) = \Psi(g).p_1(x)$
$\phi_1(g) = f_1 \in CU_n(p,q)$
f denotes the pseudo-hermitian scalar product

Fig. 3.1.

The first diagram associated with the pseudounitary conformal group $CU_n(p,q)$ (Figure 3.1) corresponds to the choice of $RU(p,q)$ as a covering group for $U(p,q)$.

The second diagram associated with the pseudounitary conformal group $CU_n(p,q)$ (Figure 3.2) corresponds to the choice of $\triangle U(p,q)$ as a covering group for $U(p,q)$.

3.9.3.5.1 Lemma (A) $p_1(x) = r(x)\delta_0' + x - \mu_0'$ *is equivalent to* $x = p_1(x) - 2R(p_1, \mu_0')\delta_0' + \mu_0'$, *if* $r(p_1(x)) = 0$ (A$_1$).



$C_{2n+4} = Q(F)$

$\mathbf{H}_{p,q} \simeq (\mathbf{R}^{2n}(2p, 2q), J)$

$\mathcal{A}'_1 \simeq \mathcal{A}_1 \times_{\mathbf{Z}_1} U(1)$ with
$\mathcal{A}_1 = \{1, -1, E_N, -E_N\}$ where
$E_N = e_0 J e_0 e_1 J e_1 \ldots e_n J e_n e_{n+1} J e_{n+1}$.

$l_1(w) = f_1$ such that $f_1(x) = \lambda_w(x)\{w.p_1(x) - 2R(p_1(x), \mu'_0)\delta'_0\} + \mu'_0$
$2\lambda_g(x) R(w.p_1(x), \mu'_0) = -1$
$\lambda_g(x) = (\sigma_g(x))^{-1}$ for $\sigma_g(x) \neq 0$
$p_1(x) = \frac{1}{2\sqrt{2}}(r(x) - 1)(e_0 + J(e_0)) + \frac{1}{2\sqrt{2}}(r(x) + 1)(e_{n+1} + J(e_{n+1})) + x$
or, equivalently, $p_1(x) = r(x)\delta'_0 + x - \mu'_0$

$\pi(g) p_1(x) g^{-1} = \sigma_g(x) p_1(f_1(x)) = \psi(g).p_1(x)$

Fig. 3.2.

All the calculations are made in the Clifford algebra $Cl^{p+1,q+1}$. R is the real part of the form f that defines the pseudo-hermitian scalar product. If $p_1(x) = r(x)\delta'_0 + x - \mu'_0$, then $p_1(x)\mu'_0 = r(x)\delta'_0\mu'_0 + x\mu'_0$ and $\mu'_0 p_1(x) = r(x)\mu'_0\delta'_0 + \mu'_0 x$, whence $p_1(x)\mu'_0 + \mu'_0 p_1(x) = 2R(p_1(x), \mu'_0) = r(x)$ and then $x = p_1(x) - 2R(p_1, \mu'_0)\delta'_0 + \mu'_0$.

We can also remark that $2R(p_1, \delta'_0) = -1$, since $p_1(x)\delta'_0 + \delta'_0 p_1(x) = 2R(p_1(x), \delta'_0) = -2R(\mu'_0, \delta'_0) = -1$.

3.9.3.5.2 Construction of the Diagram

Since φ_1 is a morphism of groups from $RU(p+1,q+1)\times_{\mathbf{Z}_2}U(1)\simeq \Delta U(p+1,q+1)$ onto $CU_n(p,q)$ and according to the classcal writing of $g\in RU(p+1,q+1)$, $\sigma_g(x)$ is a nonzero coefficient when $f_1(x)$ is defined for $f_1=\varphi_1(g)$, it follows that (A) is equivalent to (A_2):

$$p_1(f_1(x))=\lambda_g(x)\psi(g).p_1(x),\quad \text{for } f_1=\varphi_1(g).$$

Let w be in $U(p+1,q+1)$. We can associate $f_1=\varphi_1(g)\in CU_n(p,q)$ with w such that according to (A_1),

$$f_1(x)=\lambda_g(x)(w\cdot p_1(x)-2R(w\cdot p_1(x),\mu_0')\delta_0')+\mu_0'$$

with $2\lambda_g(x)R(w\cdot p_1(x),\delta_0')=-1$.

Thus, we define a surjective mapping l_1 from $U(p+1,q+1)$ into $CU_n(p,q)$. Moreover, $w\to l_1(w)=f_1=\varphi_1(g)$ is a morphism of groups. The verification of these two facts will be made in the exercises, below.

The determination of $\ker l_1$ is immediate. $f_1=\mathrm{Id}_{\mathbf{H}_{p,q}}$, with $f_1=l_1(w)$ and $w=\Psi(g)$ if and only if $g\in\mathcal{A}_1=\{1,-1,E_N,-E_N\}$, i.e., if only if $\Psi(g)=w\in\{\mathrm{Id}_{\mathbf{H}_{p+1,q+1}},-\mathrm{Id}_{\mathbf{H}_{p+1,q+1}}\}\simeq\mathbf{Z}_2$. Therefore, we have constructed an explicit algebraic isomorphism of groups X_1 from $PU(p+1,q+1)=U(p+1,q+1)/U(1)\cdot I$ onto $CU_n(p,q)$. Since the kernel of l_1 is discrete, we have also obtained an isomorphism of Lie groups. One can easily verify in the previous diagram that $l_1\circ j_1=i_1$. The verification will be made below in the exercises.

3.9.4 Pseudounitary Flat Spin Structures and Pseudounitary Conformal Flat Spin Structures

3.9.4.1 Pseudounitary Flat Spin Structures

3.9.4.1.1 A Review of Some Classical Results[32]

Let (E,Q) be a standard quadratic regular space over $\mathbf{R}$. We want to recall briefly some classical results. With our previous notation, an $RO(Q)$-spin flat structure[33] is defined

[32] Cf., for example, (I) A. Lichnerowicz, (a) Champs spinoriels et propagateurs en relativité générale, *Bull. Soc. Math. de France*, 92, pp. 11–100, 1964; (b) Champ de Dirac, champ du neutrino et transformation C.P.T. sur un espace courbe, *Ann. l'I.H.P., Section A (N.S.)*, 1, pp. 233–290, 1964. (II) Y. Choquet-Bruhat, *Géométrie Différentielle et 9 Systèmes Extérieurs*, chap. III, pp. 126–135, Dunod, Paris, 1968. (III) A. Crumeyrolle, (a) Structures spinorielles, *Ann. l'I.H.P., Section A (N.S.)*, vol. 11, no 1, pp. 19–55, 1969; (b) Groupes de spinorialité, *Ann. l'I.H.P., Section A (N.S.)*, vol. 14, no 4, pp. 309–323, 1971; (c) Fibrations spinorielles et twisteurs généralisés, *Periodica Math. Hungarica*, vol. 6.2, pp. 143–171, 1975.

[33] Cf. A. Crumeyrolle, (a) Structures spinorielles, *Ann. l'I.H.P., Section A (N.S.)*, vol. 11, no 1, pp. 19–55, 1969; (b) Groupes de spinorialité, *Ann. l'I.H.P., Section A (N.S.)*, vol. 14, no 4, pp. 309–323, 1971; (c) Fibrations spinorielles et twisteurs généralisés, *Periodica Math. Hungarica*, vol. 6.2, pp. 143–171, 1975.

by an equivalence class of $(\mathcal{R}, g)$, where $\mathcal{R}$ is an orthonormal real basis called abusively a "frame" and g an element of $RO(Q)$, with the following meaning: $(\mathcal{R}, g) \sim (\mathcal{R}', g')$ if and only if $\mathcal{R}' = \sigma\mathcal{R}$, $\psi(\gamma) = \sigma$ with $g' = \gamma g$, $g, g', \gamma \in RO(Q)$.

The choice of $(\mathcal{R}, g)$ in an equivalence class corresponds to fixing what will be called an $RO(Q)$ spinorial "frame" as origin, and isomorphic spaces of spinors can be associated with the possible choices. If we agree to use Witt basis Ω associated by the classical Witt process[34] with the real orthonormal basis, according to Crumeyrolle,[35] we can also define such an $RO(Q)$-spin trivial structure by an equivalence class of (Ω, g) such that $(\Omega, g) \sim (\Omega', g')$ means that $\Omega' = \sigma\Omega$, $\psi(\gamma) = \sigma$, $g' = \gamma g$, $\gamma \in RO(Q)$ and $g, g' \in RO'(Q) \simeq RO(Q')$, where the sign $'$ indicates complexification. In such a class there are always "real" Witt bases, since $O(Q')$ acts transitively on the set of real or complex Witt bases of the standard space. A trivial spin structure can thus be defined from a "nonreal" Witt basis chosen as the "origin."

3.9.4.1.2 $RU(p, q)$ Standard Spin Structures

Let us recall the fundamental exact sequence

$$1 \to Z_2 \to RU(p, q) \to U(p, q) \to 1,$$

"using" the Clifford algebra $Cl^{p,q}$, and the other one,

$$1 \to U(1) \to RU(p, q) \times_{\mathbf{Z}_2} U(1) \to U(p, q) \to 1,$$

using the Clifford algebra $C'_{2n}(2p, 2q) = (C(\mathbf{R}^{2n}, Q_{2p,2q}))'$. If we agree to choose the first covering space, spin flat pseudounitary geometry appears naturally as a particular case of structures recalled in 3.9.4.1.1 and in the second case spin flat pseudounitary geometry appears naturally as a special case of $RO'(2p, 2q)$-spin structures.

3.9.4.2 Pseudounitary Conformal Flat Spin Structures

The results of 2.5.2 can be immediately applied with the following changes:

$$n \to 2n,\ p \to 2p,\ q \to 2q,\ r \to 2r,\ e_N \to E_N.$$

Thus, we can give the following definitions.

3.9.4.2.1 Definition A conformal pseudounitary spinor of $\mathbf{H}_{p,q}$, associated with a complex representation ρ of $RU(p+1, q+1)$ in a space of spinors for the Clifford algebra $(Cl^{p+1,q+1})'$ is by definition an equivalence class $((\tilde{\mathcal{R}}_{2n+2})_1, g, X_{2n+2})$, where $(\tilde{\mathcal{R}}_{2n+2})$ is a projective "pseudounitary frame" of $P(\mathbf{H}_{p+1,q+1})$, $g \in RU(p+1, q+1)$, $X_{2n+2} \in \mathbf{C}^{2^{2r+1}}$ and where $((\tilde{\mathcal{R}}'_{2n+2})_1, g', X'_{2n+2})$ is equivalent to $((\tilde{\mathcal{R}}_{2n+2})_1, g, X_{2n+2})$ if and only if we have $\tilde{\mathcal{R}}'_{2n+2} = \sigma(\tilde{\mathcal{R}}_{2n+2})$, $\sigma = \eta_1(\gamma) \in PU(p+1, q+1)$ with $\gamma = g' g^{-1}$ and $X'_{2n+2} = {}^t(\rho(\gamma))^{-1} X_{2n+2}$, where $(\rho(\gamma))^{-1}$ is identified with an endomorphism of $\mathbf{C}^{2^{2r+1}}$.

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[34] Cf., for example, C. Chevalley, *The Algebraic Theory of Spinors*, op. cit., chapter 1, pp. 8–19.
[35] A. Crumeyrolle, *Structures Spinorielles*, op. cit., p. 35.

3.9.4.2.2 Definition We define an equivalence class $((\tilde{\mathcal{R}}_{2n+2})_1, g)$, where g is in $RU(p+1, q+1)$ and $((\tilde{\mathcal{R}}_{2n+2})_1)$ is a projective orthogonal frame of $PU(p+1, q+1)$, to be a "conformal pseudounitary frame" of $\mathbf{H}_{p,q}$ associated with the "real" orthonormal base $(B_1)_n$ of $\mathbf{H}_{p,q}$. $((\tilde{\mathcal{R}}_{2n+2})_1, g)$ is equivalent to $((\mathcal{R}_{n+2})_1, g')$ if and only if $(\tilde{\mathcal{R}}_{2n+2})_1 = (\tilde{\mathcal{R}}_{2n+2})_1\sigma$ and $\sigma = \eta_1(\gamma)$, where $g, g' \in RU(p+1, q+1)$ and $\gamma = g'g^{-1}$.

We remark that

$$((\tilde{\mathcal{R}}_{2n+2})_1, g) \sim ((\mathcal{R}_{2n+2})_1, -g) \sim ((\mathcal{R}_{2n+2})_1, E_N g) \sim ((\mathcal{R}_{2n+2})_1, -E_N g).$$

If we suppose g, g' in $RO'(2n+2)$ and $\gamma = g'g^{-1} \in RU(p+1, q+1)$, we can consider the action of $RU(p+1, q+1)$ on every pseudounitary spinor frame of $\mathbf{C}'_{2n+2} f_{2r+1}$.

3.9.4.2.3 Definition With obvious notation, $(\tilde{\Omega}_{2n+2})_1$ and $(\tilde{\Omega}'_{2n+2})_1$ being projective orthogonal Witt frames of $PU(p+1, q+1)$, $((\tilde{\Omega}_{2n+2})_1, g)$ and $((\tilde{\Omega}'_{2n+2})_1, g)$ define the same flat conformal pseudounitary spin structure if and only if $(\tilde{\Omega}_{2n+2})_1 = \sigma(\tilde{\Omega}_{2n+2})_1$, $\eta'_1(\gamma) = \sigma$, $\gamma = g'g^{-1}$, $g, g' \in (RU(p+1, q+1))'$, $\gamma \in RU(p+1, q+1)$.

Thus,

$$((\tilde{\Omega}_{2n+2})_1, g) \sim ((\tilde{\Omega}_{2n+2})_1, -g) \sim ((\tilde{\Omega}_{2n+2})_1, E_N g) \sim ((\tilde{\Omega}_{2n+2})_1, -E_N g).$$

3.9.5 Study of the Case $n = p + q = 2r + 1$

If $n = 2r + 1$, then $2n = 2r + 2$, and we can consider a special Witt decomposition of E'_{2n} that leads to analogous conclusions.

3.10 Pseudounitary Spinoriality Groups and Pseudounitary Conformal Spinoriality Groups

3.10.1 Classical Spinoriality Groups (cf. 3.13 Appendix below)

This notion is due to Albert Crumeyrolle.[36] Let (E, Q) be a standard $2r$-dimensional quadratic regular space over $\mathbf{R}$. A. Crumeyrolle introduces the Clifford regular group $\tilde{G}$ and the groups Pin Q and Spin Q defined respectively as the multiplicative group of elements g in $\tilde{G}$ such that $|N(g)| = 1$, where N is the graduate norm of g and Spin Q = Pin $Q \cap C^+(Q)$. Let $\{x_i, y_j\}_{1 \le j \le r}^{1 \le i \le r}$ be a "real" Witt base of $\mathbf{C}^n$ associated with an orthonormal base of $\mathbf{R}^n$. Let $f = y_1 \cdots y_r$ be an r-isotropic vector associated with a maximal totally isotropic subspace.

[36] Cf., for example, A. Crumeyrolle, Spin fibrations over manifolds and generalized twistors, *Proceedings of Symposia in Pure Mathematics*, vol. 27, pp. 53–67, 1975.

3.10.1.1 Definition H is the subgroup consisting of elements $\gamma \in \mathrm{Spin}\, Q$ such that $\gamma f = \pm f$. $\psi(H) = \mathcal{G}$ is the spinoriality group associated with f.

3.10.1.2 Proposition *In elliptic signature, $\mathcal{G}$ can be identified with $SU(r, \mathbf{C})$; $\mathcal{G}$ is the set of elements with determinant* 1 *in the stabilizer of a maximal totally isotropic subspace,* $\dim \mathcal{G} = r^2 - 1$. *$\mathcal{G}$ is connected and simply connected.*

3.10.1.3 Proposition *In signature $(k, n-k)$, $k < n-k$, k positive terms and $r \geq 2$, $\mathcal{G}$ is isomorphic to the subgroup of determinant* 1 *with matrix*

$$\begin{pmatrix} \alpha & -\bar{\mu} & \lambda & \mu \\ 0 & \beta & \nu & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & -\bar{\nu} & \bar{\beta} \end{pmatrix}$$

with $\alpha \in M_k(\mathbf{R})$, $\det \alpha = \pm 1$, $\beta \in M_{r-k}(\mathbf{C})$, $\beta^t \bar{\beta} = \mathrm{Id}$, $\det \beta = \pm \det \alpha$, $\alpha^t \rho = \mathrm{Id}$, $\lambda \in M_k(\mathbf{R})$, $\mu \in \mathbf{C}^{k(r-k)}$, $\nu \in \mathbf{C}^{(r-k)k}$, $\nu = -\beta^t \mu \rho$, ${}^t\rho\lambda + {}^t\lambda\rho = {}^t\nu\bar{\nu} + {}^t\bar{\nu}\nu$. *$\mathcal{G}$ has four connected components and can be identified with the set of elements with determinant* 1 *in the stabilizer of a maximal totally isotropic subspace* $\dim \mathcal{G} = r^2 - 2 + k(k-1)/2$.

3.10.1.4 Proposition *If Q is a neutral form $(k = r)$, $\mathcal{G}$ is isomorphic to the subgroup of elements in $SL(n, \mathbf{R})$ with the matrix $\left(\begin{smallmatrix} 0 & \lambda \\ 0 & \rho \end{smallmatrix}\right)$ such that $\alpha \in M_r(\mathbf{R})$,* $\det \alpha = 1$, $\alpha^t \rho = \mathrm{Id}$, ${}^t\rho\lambda + {}^t\lambda\rho = 0$. *$\mathcal{G}$ is connected and can be identified with the set of elements with determinant* 1 *in the stabilizer of a maximal totally isotropic subspace;* $\dim \mathcal{G} = (r-1)(3r+2)/2$.

3.10.1.5 Definition Let $H_e = \{\gamma \in \mathrm{Spin}\, Q : \gamma f = X e^{i\theta} f, X \in \mathbf{R}^*\}$. We call $\psi(H_e) = \mathcal{G}_e$ the enlarged spinoriality group associated with f. $\mathcal{G}_e$ is the stabilizer of a maximal totally isotropic subspace for the action of $SO(Q)$. In elliptic signature $X = 1$ and $\psi(H_e) = \mathcal{G}_e \simeq U(r, \mathbf{C})$. In general, $X \neq 1$. Such subgroups of $SO(Q) : \mathcal{G}_e$ satisfy $\dim \mathcal{G}_e = r^2 + k(k-1)/2$ for any $k : 0 \leq k \leq r$. If $k \neq 0$, such a group $\mathcal{G}_e$ is not a generalized unitary group. $\mathcal{G}_e$ is connected if $k = 0$ and has 2 connected components if $k \neq 0$. $\mathcal{G}$ is an invariant subgroup of $\mathcal{G}_e$. Both $\mathcal{G}$ and $\mathcal{G}_e$ are associated with the same isotropic r-vector (cf. exercises below).

3.10.2 Pseudounitary Spinoriality Groups

3.10.2.1 Definition Let H, respectively H_e, be the subgroup of elements $\gamma \in \mathrm{Spin}\, Q(2p, 2q)$ such that $\gamma f_{2r} = \pm f_{2r}$, respectively $\gamma f_{2r} = X e^{i\theta} f_{2r}$, with $X \in \mathbf{R}^*$. We call $\psi(H) = \mathcal{G}$, respectively $\psi(H_e) = \mathcal{G}_e$, the pseudounitary group, respectively the enlarged spinoriality group, associated with f_{2r}.

3.10.2.2 Characterization

Characterizations of these groups are immediate following 3.10.1. For example, $\mathcal{G}_e$ appears as the stabilizer of the maximal totally isotropic subspace associated with $f_{2r} = y_1 \cdots y_{2r}$, the considered $2r$-isotropic vector (cf. exercises below).

3.10.3 Pseudounitary Conformal Spinoriality Groups

3.10.3.1 Review

Let us recall that we consider $SO(2p, 2q)$ and the exact sequence

$$1 \to Z_2 \to RU(p, q) \to U(p, q) \to 1,$$

and that (δ'_0, μ'_0) constitutes an isotropic basis of $\mathbf{H}(1, 1)$. As previously, we can consider an isotropic $(2r + 1)$ vector in $(\mathbf{R}^{2n}(2p, 2q)' \oplus \mathbf{H}_{1,1}$ of the form $f_{2r+1} = y_1 \cdots y_{2r}\mu'_0$. The mapping p_1 from $\mathbf{H}_{p,q}$ into $F = \mathbf{H}_{p,q} \oplus \mathbf{H}_{1,1}$ such that $p_1(x) = r(x)\delta'_0 + x - \mu'_0$ has previously been introduced (cf. 3.3.1).

3.10.3.2 Definition Let $\mathcal{A}_1 = \{1, -1, E_N, -E_N\}$. Let $(H_{UC})_e = \{\gamma, \gamma \in RU(p{+}1, q + 1) : \gamma f_{2r+1} = \mu f_{2r+1}$, where $\mu \in \mathbf{C}^*\}$. By definition, we call any subgroup $(S_{UC})_e = \varphi_1(H_{UC})_e$ an enlarged pseudounitary conformal spinoriality group associated with f_{2r+1} . Let $(H_{UC}) = \{\gamma, \gamma \in RU(p{+}1, q{+}1) : \gamma f_{2r+1} = \varepsilon_1 f_{2r+1}$, where $\varepsilon_1 \in \mathcal{A}_1\}$. By definition, we call any subgroup $S_{UC} = \varphi_1(H_{UC})$ of $CU_n(p, q)$ a pseudounitary conformal spinoriality group in a strict sense. According to 3.9.3.4 such a definition is equivalent to the following: H_C is the set of element $\gamma \in RU(p{+}1, q{+}1)$ such that $\gamma f_{2r+1} = \pm f_{2r+1}$.

3.10.3.3 Characterizations of Enlarged Pseudounitary Conformal Spinoriality Groups

Since $p_1(x) = r_1(x)\delta'_0 + x - \mu'_0$, for any $i, 1 \leq i \leq 2r$, we remark that $p_1(y_i) = y_i - \mu'_0$ and $p_1(0) = -\mu'_0$.

Up to this light change of notations, the demonstration given in 2.5.1.4 can be applied.

If $f_1 = l_1(w)$ with $w \in U(p + 1, q + 1)$, as, in 2.5.1.4, we can verify that $f_1(y_1) \dots f_1(y_{2r}) = \mu y_1 \dots y_{2r}$, where $\mu \in \mathbf{C}^*$.

Thus we have obtained the following proposition.

3.10.3.4 Proposition *The enlarged pseudounitary conformal spinoriality group is the stabilizer of the maximal totally isotropic subspace F' associated with the $2r$-isotropic vector $y_1 \cdots y_{2r}$.*

3.10.3.5 Characterization of Pseudounitary Conformal Spinoriality Groups in a Strict Sense

Normalization conditions appear: μ is equal to ± 1. A more precise study will be given below in the exercises.

3.11 Pseudounitary Spin Structures on a Complex Vector Bundle

3.11.1 Review of Complex Pseudo-Hermitian Vector Bundles

A general theory of real or complex vector bundles is given for example in the following bibliography.[37]

3.11.1.1 Definition A pseudo-hermitian complex vector bundle, denoted by (ξ, η) or simply ξ, is a differentiable complex vector bundle over a differentiable manifold M (that can be paracompact, real or complex, or almost complex or almost pseudo-hermitian), with typical fiber the standard pseudo-hermitian space $H_{p,q}$ equipped with the pseudo-hermitian sesquilinear nondegenerate form η on its fibers, that varies differentiably from fiber to fiber.

3.11.1.2 Definition Let us assume that the manifold M is an almost complex manifold. If R_ξ denotes the principal bundle of "linear frames of ξ," the structure group $GL(n, \mathbf{C})$ can be reduced to $U(p, q)$, $p + q = n$. We agree to denote by U_ξ the principal bundle of orthogonal normalized bases suitable for the almost pseudo-hermitian structure. The associated bundle $U_\xi \times_{U(p,q)} \mathbf{H}_{p,q}$ is isomorphic to ξ. We agree to denote the isomorphic class of U_ξ or of $(\xi, \tilde{\eta})$ by $[U_\xi]$. For an almost pseudo-hermitian manifold $(M, \tilde{\eta})$ one can define the tangent bundle $\xi = T(M)$ and the cotangent bundle $\xi_1 = T^*(M)$. R_M is then the "bundle of linear frames" and U_M the bundle of normalized orthogonal bases suitable for the pseudo-hermitian structure, which is called the almost pseudo-hermitian bundle.

3.11.2 Pseudounitary Spin Structures on a Complex Vector Bundle

3.11.2.1 General Definition of a Spin Classical Structure

The notion of spin structure on a manifold V has been introduced by A. Haefliger, who made specific an idea from Ehresmann.[38] Many authors such as J. Milnor, A. Lichnerowicz, R. Deheuvels, I. Popovici, W. Greub, B. Kostant, M. L. Michelson, R. Coquereaux, A. Jadczyk, have taken an interest in the study of those structures.

3.11.2.1.1 Definition (General Definitions) Let ξ be a pseudo-Euclidean real vector bundle (ξ, Q) or respectively a complex vector bundle (ξ_1, Q_1). By definition a spin structure defined on such a bundle is any lifting of the corresponding principal bundle

[37] Cf., for example, D. Husemoller, *Fibre Bundles*, op. cit.; N. Steenrod, *The Topology of Fibre Bundles*, op. cit.; S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, op. cit.; A. L. Besse, *Manifolds All of Whose Geodesics Are Closed*, op. cit.

[38] A. Haefliger, Sur l'extension du groupe structural d'un espace fibré, *C.R. Acad. Sci., Paris*, 243, pp. 558–560, 1956.

O_ξ, respectively O_{ξ_1}, for the corresponding covering morphisms associated with the group $O(Q)$, respectively $O(Q_1)$.

3.11.2.1.2 Example A spin structure on a pseudo-Euclidean real vector bundle (E, Q), i.e., by definition an $RO(Q)$-structure, is a principal bundle morphism from RO_ξ into O_ξ, where RO_ξ denotes the lifting of the bundle O_ξ associated with the morphism φ of groups $RO(Q) \to O(Q)$. The complex case can be defined in the same pattern. Such a definition can be naturally extended to the case of pseudo-riemannian manifolds and leads to the notion of spin structures on the tangent bundle or cotangent bundle of M.

3.11.3 Obstructions to the Existence of Spin Structures[39]

We use the notation of Greub and Petry.[40] The obstructions to the existence of spin structures can be viewed in the general theory of lifting structure groups.

3.11.3.1 Proposition *Thus the condition that $w_2[\xi, \varphi] = K[O_\xi, \varphi]$ vanishes, with the notation of Greub and Petry[40] is equivalent to the existence of an $RO(Q)$-spin structure on the pseudo-Euclidean real vector bundle (ξ, Q), where φ is the covering morphism of groups $\varphi : RO(Q) \to O(Q)$ and $w_2[\xi, \varphi]$ is the second Stiefel–Whitney class of (ξ, Q).*

3.11.4 Definition of the Fundamental Pseudounitary Bundle

3.11.4.1 Definition of a Pseudounitary Spin Structure

By definition, a pseudounitary spin structure on a vector bundle (ξ, η)-pseudo-hermitian complex vector bundle is a lifting of the principal bundle U_ξ relative to the associated morphism δ of groups $RU(p, q) \times_{\mathbf{Z_2}} U(1) \xrightarrow{\delta} U(p, q)$.

Such a definition can be extended to the case of manifolds with almost pseudo-hermitian structure via the introduction of the tangent bundle and the cotangent bundle.

3.11.4.2 Notation We denote by RU_ξ the principal bundle associated with the principal bundle U_ξ for the corresponding morphism $\delta : RU(p, q) \times_{\mathbf{Z_2}} U(1) \to U(p, q)$.

3.11.4.3 Remark We can also introduce the principal bundle $R_1 U_\xi$, the lift of the principal bundle U_ξ relative to the morphism $\psi : RU(p, q) \to U(p, q)$ introduced in the fundamental diagram.

[39] Cf., for example, W. H. Greub, S. Halperin, R. Vanstone, *Connections, Curvature and Cohomology*, op. cit.; W. Greub and R. Petry, *On the Lifting of Structure Groups*, op. cit.; A. Haefliger, op. cit.; F. Hirzebruch, *Topological Methods in Algebraic Geometry*, op. cit.; M. Karoubi, *Algèbres de Clifford et K-Théorie*, op. cit.; B. Kostant, *Quantization and Unitary Representations*, op. cit.; M. L. Michelson, Clifford and spinor cohomology, *Amer. J. Math.*, vol. 106, no 6, 1980, pp. 1083–1146; J. Milnor, *Spin Structure on Manifolds*, op. cit.

[40] W. Greub and R. Petry, *On the Lifting of Structure Groups*, op. cit., pp. 219–241, see p. 242.

3.11.4.4 Definition of the Fundamental Pseudounitary Bundle

According to 3.7.5.2, if we consider a pseudounitary spin structure $\delta : RU_\xi \to U_\xi$, we agree ro call, by definition, the bundle $\alpha'(RU_\xi)$ extension of the bundle RU_ξ for the morphism α' of the diagram the fundamental pseudounitary bundle. Such a principal bundle $\alpha'(RU_\xi)$ has for structure group $U(1)$. Its associated complex vector bundle is $RU_\xi \times_{\alpha'} \mathbf{C}$, denoted by α'_{RU_ξ}.

3.11.4.5 Existence of Pseudounitary Spin Structures

3.11.4.5.1 Preliminaries

We have found two exact sequences:

$$1 \to Z_2 \to RU(p,q) \xrightarrow{\Psi} U(p,q) \to 1$$

and

$$1 \to U(1) \to RU(p,q) \times_{\mathbf{Z}_2} U(1) \xrightarrow{\delta} U(p,q) \to 1.$$

For the principal bundle U_ξ we introduce naturally two liftings of structures $R_1 U_\xi$ for ψ and RU_ξ relative to δ. We will use the fundamental general results of Greub and Petry.[41]

3.11.4.5.2 Definition (Theorem) Let $P = (P, \pi, B, G)$ be a principal bundle, where P and B are topological spaces and G is a topological group. Let $\rho : \Gamma \to G$ be a continuous homomorphism from a topological group Γ onto G with kernel K. ρ will be called central if

(i) K is discrete,
(ii) K is contained in the center of Γ (thus, in particular, K is abelian).

A Γ-structure on P is a Γ-principal bundle $\tilde{P} = (\tilde{P}, \tilde{\pi}, B, \Gamma)$ together with a strong bundle map $\eta : \tilde{P} \to P$ that is equivariant under the right actions of the structure groups; that is $\eta(\tilde{z} \cdot \gamma) = \tilde{\eta}(\tilde{z}) \cdot \rho(\gamma)$ for any $\tilde{z} \in \tilde{P}$, $\gamma \in \Gamma$. We will choose an open covering $U = \{U_i\}$ of B such that P is trivial over every U_i. We assume also that B is an L-space, that is, by definition, that every open covering has a simple refinement, i.e., such that all the non-empty intersections $U_{i_1} \cap \ldots \cap U_{i_p}$ are contractible. (cf. below exercises).

There exists an element $K(P, \rho)$, called the Γ-obstruction class, of $\check{H}^2(B, K)$, the second Čech cohomology group of B with coefficients in K, with the following fundamental property: P admits a Γ-structure if and only if $K(P, \rho) = e$.

The demonstration will be given below in the exercises.

We recall the following statements already used, and which will be applied later.

[41] W. Greub and N.R. Petry, *On the Lifting of Structure Groups*, op. cit., pp. 217–246.

3.11.4.5.3 λ-Extensions

Let $P = (P, \pi, B, G)$ be a principal G-bundle and let $\lambda : G \to G'$ be a homomorphism. Then λ determines a principal bundle $P_\lambda = (P_\lambda, \pi_\lambda, B, G')$ over the same base in the following way: Choose a covering U_i of B with a system of local sections σ_i and transition functions g_{ij}. Define maps

$$g'_{ij} = \lambda \circ g_{ij}.$$

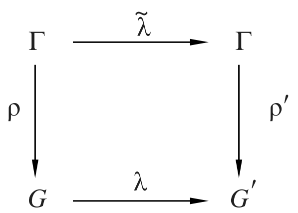
Then the g'_{ij} satisfy the relation

$$g'_{ij}(x)g'_{jk}(x)g'_{ki}(x) = e,$$

and consequently, there is a principal bundle P_λ with a system of local sections such that the g'_{ij} are the corresponding transition functions. P_λ is called the **λ-extension** of P.

3.11.4.5.4 Definition Next assume that $\rho : \Gamma \to G$ and $\rho' : \Gamma' \to G'$ are central homomorphisms with kernels K and K' respectively. Let $\lambda : G \to G'$ be a homomorphism. We say that ρ and ρ' are related if there is a continuous map $\tilde{\lambda}$ (not necessarily a homomorphism) such that

(1) the following diagram commutes and



(2) $\tilde{\lambda}(k \bullet g) = \tilde{\lambda}(k)\tilde{\lambda}(g)$ for all $k \in K$, $g \in \Gamma$.[42]

$\tilde{\lambda}$ restricts to a homomorphism of K into K'. In fact, if $k \in K$ then $\rho'\tilde{\lambda}(k) = \lambda \bullet \rho(k) = \lambda(e) = e'$, whence $\tilde{\lambda}(k) \in K'$. Next let P_λ be a λ-extension of P and assume that the homomorphisms $\rho : \Gamma \to G$ and $\rho' : \Gamma' \to G'$ are related. Choose a simple covering $\{U_i\}$ of B and let γ_{ij} be a lifting of g_{ij}. Set

$$\gamma'_{ij} = \tilde{\lambda} \bullet \gamma_{ij}.$$

Then

$$\rho'\gamma'_{ij} = g'\tilde{\lambda}\gamma_{ij} = \lambda\rho\gamma_{ij} = \lambda g_{ij} = g'_{ij},$$

and so the γ'_{ij} are liftings of the g'_{ij}.

42 W. Greub and N. R. Petry, op. cit., p. 222.

Now set

$$\theta_{ijk}(x) = \tilde{\lambda}(\gamma_{ij}(x))\tilde{\lambda}(\gamma_{jk}(x))\tilde{\lambda}(\gamma_{ij}(x)\gamma_{jk}(x))^{-1}, \quad x \in U_{ijk}.$$

Then we have

$$\begin{aligned}\rho'\theta_{ijk}(x) &= \lambda\rho(\gamma_{ij}(x))\lambda\rho(\gamma_{jk}(x)) \bullet \lambda\rho(\gamma_{ij}(x))\gamma_{jk}(x)^{-1} \\ &= \lambda(g_{ij}(x))\lambda(g_{ik}(x))\lambda(g_{ij}(x)g_{ik}(x))^{-1} \\ &= \lambda\left[g_{ij}g_{ik}(g_{ij}g_{ik})^{-1}\right] = \lambda(e) = e'.\end{aligned}$$

It follows that

$$\theta_{ijk}(x) \in K', x \in U_{ijk},$$

and so these functions are constant. Hence they define a 2-cochain θ in the nerve $N(U)$ with values in K'. (See below appendix for the definition of the nerve.)

3.11.4.5.5 Proposition *Let p and p' denote the 2-cocycles for P and P_λ obtained via the liftings γ_{ij} and γ'_{ij}. Then*

$$p'(i, j, k) = \theta(i, j, k)\tilde{\lambda}p(i, j, k).$$

Proof. Applying $\tilde{\lambda}$ to the equation

$$\gamma_{ik} = p_{ijk}^{-1}\gamma_{ij}\gamma_{jk}$$

and using (2), we obtain

$$\gamma'_{ik} = \tilde{\lambda}(p_{ijk}^{-1})\tilde{\lambda}(\gamma_{ij}\gamma_{jk}) = \tilde{\lambda}(p_{ijk})^{-1}\tilde{\lambda}(\gamma_{ij}\gamma_{jk}).$$

On the other hand,

$$\gamma'_{ik} = p'^{-1}_{ijk}\gamma'_{ij}\gamma'_{jk} = p'^{-1}_{ijk}\tilde{\lambda}(\gamma_{ij})\tilde{\lambda}(\gamma_{jk}).$$

These equations yield

$$p'^{-1}_{ijk}\tilde{\lambda}(\gamma_{ij})\tilde{\lambda}(\gamma_{jk}) = \tilde{\lambda}(p_{ijk})^{-1}\tilde{\lambda}(\gamma_{ij}\gamma_{jk}),$$

whence

$$p'^{-1}_{ijk} = \tilde{\lambda}(p_{ijk})^{-1}\tilde{\lambda}(\gamma_{ij}\gamma_{jk})\tilde{\lambda}(\gamma_{jk})^{-1}\tilde{\lambda}(\gamma_{ij})^{-1}.$$

It follows that

$$p'_{ijk} = \tilde{\lambda}(\gamma_{ij})\tilde{\lambda}(\gamma_{jk})\tilde{\lambda}(\gamma_{ij}\gamma_{jk})^{-1}\tilde{\lambda}(p_{ijk}) = \theta(i, j, k)\tilde{\lambda}(p_{ijk}).$$

Since p and p' are cocycles and since the restriction of $\tilde{\lambda}$ to K is a homomorphism, it follows from the lemma that θ is a cocycle. Thus it represents an element $\theta \in \check{H}^2(B, K')$. Now we have as an immediate consequence of the proposition the following.

3.11.4.5.6 Corollary *The obstruction classes of P and P_λ are connected by the relation*

$$\kappa(P_\lambda) = \theta \bullet \tilde{\lambda}_* \kappa(P),$$

where

$$\tilde{\lambda}_* = \check{H}(B, K) \to \check{H}(B, K')$$

denotes the homomorphism induced by the homomorphism $\tilde{\lambda} : K \to K'$. In particular, if $\tilde{\lambda}$ is a homomorphism of Γ, then

$$\kappa(P_\lambda) = \tilde{\lambda}_* \kappa(P).$$

3.11.4.5.7 Bundles with Structure Group $O(p, q)$. Proposition

Let $P = (P, \pi, B, O(p, q))$ be an $O(p, q)$ bundle. There exist two fundamental classes $K_1(P) \in \check{H}^1(B, \mathbf{Z}_2)$ and $K(P)$ such that P admits an $RO(p, q)$-spin structure if and only if $K(P) = 0$. $K_1(P)$ and $K(P)$ coincide respectively with the first and the second Stiefel–Whitney classes w_1 and w_2 of P.

This important characterization is due to Greub and Petry, On the lifting of structure groups, op. cit., pp. 240–242. See also Max Karoubi, Algèbres de Clifford et K-théorie, op. cit., Proposition 1.1.26, p. 174, and Proposition 1.1.27, pp. 175–176.

3.11.4.6 Necessary and Sufficient Condition for the Existence of a Pseudounitary Spin Structure

Let us consider now a $U(p, q)$ bundle P. The application of Theorem 3.11.4.5.2 leads us to the following statement:

3.11.4.6.1 Theorem *A $U(p, q)$ bundle P admits a pseudounitary spin structure if and only if the class of obstruction $K(P, \psi)$ vanishes, or equivalently, the class of obstruction $K(P, \delta)$ vanishes (with previous notation).*

3.12 Pseudounitary Spin Structures and Pseudounitary Conformal Spin Structures on an Almost Complex 2*n*-Dimensional Manifold *V*

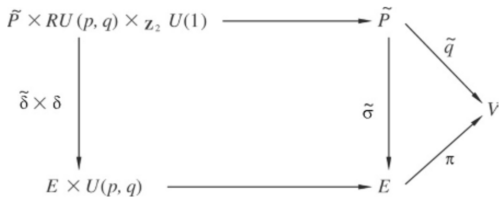
3.12.1 Pseudounitary Spin Structures

Let V be a $2n$-dimensional almost complex manifold. The definitions given in Section 3.11 lead to the following one. Let $\xi(E, V, U(p, q), \pi)$ be the bundle of the normalized orthogonal basis suitable for the pseudo-hermitian structure.

3.12.1.1 Definition V admits a pseudounitary spin structure in a strict sense if there exists a principal fiber bundle $S = (P, V, RU(p,q), q')$ and a principal morphism $\tilde{\psi}$ from S onto ξ such that the following diagram is commutative, where the horizontal mappings represent right translations:

Such a definition translates to the lifting of the bundle ξ corresponding to the morphism $\tilde{\psi}$ from S onto ξ and to the morphism of the group ψ from $RU(p,q)$ onto $U(p,q)$. We give also the following equivalent definition associated with the morphism of the covering group $\delta : RU(p,q) \times_{\mathbf{Z}_2} U(1) \to U(p,q)$.

3.12.1.2 Definition V admits a pseudounitary spin structure in a strict sense if there exists a principal fiber bundle $\tilde{S} = (\tilde{P}, V, RU(p,q) \times_{\mathbf{Z}_2} U(1), \tilde{q})$ and a principal morphism $\tilde{\delta}$ from $\tilde{S}$ onto ξ such that the following diagram is commutative, where the horizontal mappings represent right translations:



3.12.1.3 Definition In each case S, respectively $\tilde{S}$, is called the principal bundle of "spin frames" of V. And we associate σ, respectively $\tilde{\sigma}$, with S, respectively $\tilde{S}$, where $\sigma = (P \times_{RU(p,q)} \mathbf{C}^{2^{2r}}, V, RU(p,q), \mathbf{C}^{2^{2r}})$, the complex vector bundle of dimension 2^{2r}, with the typical fiber $\mathbf{C}^{2^{2r}}$ is called the bundle of pseudounitary spinors, respectively $\tilde{\sigma} = (P \times_{RU(p,q) \times_{\mathbf{Z}_2} U(1)} \mathbf{C}^{2^{2r}}, V, RU(p,q) \times_{\mathbf{Z}_2} U(1), \mathbf{C}^{2^{2r}})$ is called the bundle of pseudounitary spinors. According to 3.7.7.4 we know that the dimension of a space of spinors is $2^n = 2^{2r}$.)

We chose the definition given in 3.12.1.1. All the calculations are made in the real Clifford $Cl^{p,q}$, which is included in $C^+_{2p,2q}$. We can now introduce the following propositions,[43] the proofs of which will be given in the exercises below.

[43] The pattern is the same as that introduced by A. Crumeyrolle in Fibrations spinorielles et twisteurs généralisés, *Periodica Math. Hungarica*, vol. 6.2, pp. 143–171, 1975.

3.12.2 Necessary Conditions for the Existence of a Pseudounitary Spin Structure in a Strict Sense on *V*

The accent $'$ indicates complexification.

3.12.2.1 Proposition (i) *If there exists on V a pseudounitary spin structure in a strict sense, there exists on V, modulo a factor equal to ± 1, an isotropic $2r$-vector field, pseudo–cross section of the bundle $Cli(V, Q'_{2p,2q})$ and then a subfibering of $Cli(V, Q')$.*[44]

(ii) *The complexified bundle $\xi_{\mathbf{C}}$ admits local cross sections over the trivializing open sets (U_α) with transition functions $\psi(g_{\alpha\beta})$, $g_{\alpha\beta}(x) \in RU(p,q)$ such that if $x \in U_\alpha \cap U_\beta \to f_{\alpha\beta}$ defines locally the isotropic $2r$-vector field defined above, then $f_\beta(x) = N(g_{\alpha\beta}(x)) f_\alpha(x)$, where $f_\beta(x) = \hat{g}_{\alpha\beta}(x) f_\alpha(x) \hat{g}_{\alpha\beta}^{-1}(x)$, where $\hat{g}_{\alpha\beta}(x) = \mu_\alpha^x(x_i, y_j)$, μ being the linear isomorphism that leads to the identification of $C(Q'_{2p,2q})$ with $\mathbf{C}^{2^{2n}}$ and μ_α^x the isomorphism from $C(Q'_{2p,2q})$ onto $C(Q')_x$, where $C(Q')_x$ is the Clifford algebra induced by $C(Q')$ at x.*[45]

(iii) *The structure group of the bundle ξ can be reduced in $O(Q')$ to a pseudounitary group in a strict sense.*

3.12.3 Sufficient Conditions for the Existence of a Pseudounitary Spin Structure in a Strict Sense on *V*

3.12.3.1 Proposition *Let $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ be a trivializing atlas for the bundle $\xi_{\mathbf{C}}$ on V, with transition functions $\psi(g_{\alpha\beta}(x)) \in O(Q_{2p,2q})$. If there exists over V an isotropic $2r$-vector pseudo-field pseudo-cross section in $Cli(V, Q'_{2p,2q})$ locally defined by $x \in U_\alpha \to f_\alpha(x)$ such that for $x \in U_\alpha \cap U_\beta \neq \emptyset$, we have*

$$f_\beta(x) = \hat{g}_{\alpha\beta} f_\alpha(x) \hat{g}_{\alpha\beta}^{-1}(x),\ \mu_\alpha^x(g_{\alpha\beta}(x)) = \hat{g}_{\alpha\beta} f_\alpha(x),$$

then the manifold V admits a pseudounitary spin structure in a strict sense.

3.12.3.2 Proposition *Let us assume that the structure group of the bundle ξ reduces in $O(Q'_{2p,2q})$ to a pseudounitary spinoriality group. Then the manifold V admits a pseudounitary spin structure in a strict sense.*

---

[44] If (V, Q) is a real n-dimensional pseudo-riemannian manifold, the Clifford bundle $Cli(V, Q)$ is defined as follows: The action of the group $O(Q)$ on $\mathbf{R}^n$ can be extended to $C(\mathbf{R}^n, Q)$ by an easy verification of the fact that $O(Q)$ conserves the two-sided ideal generated by $\{x \otimes x - Q(x) \cdot 1\}$. By definition we associate a vector bundle with typical fiber $C(Q)$ and structure group the extension of $O(Q)$ introduced above with the principal bundle of orthonormal basis of V. This bundle, denoted by $Cli(V, Q)$, is of rank 2^n on $\mathbf{R}$ and its fibers are real Clifford algebras.

[45] Classically the Clifford algebra $C_V(Q)$ of V is the quotient of the tensor algebra $\otimes D^1(V)$ of differentiable vector fields on V by the two-sided ideal J generated by the elements $X \otimes V - Q(X)1$, $X \in D^1(V)$. The algebra $C_V(Q)$ induces naturally at any $x \in V$ a Clifford algebra $C(Q)_x$ or $C_x(Q)$—2^n-dimensional algebra quotient of $\otimes T_x$, where T_x is the tangent space to V at x—by the two-sided ideal generated by the elements $X_x \otimes X_x - Q(X_x) \cdot 1$.

3.12.4 Manifolds V With a Pseudounitary Spin Structure in a Broad Sense

Let f_{2r} be an isotropic $2r$-vector of $\mathbf{H}_{p,q}$. In 3.10.3.2 and 3.10.3.4 we defined the enlarged pseudounitary spinoriality group $(S_u)_e$ as the stabilizer of the maximal totally isotropic subspace F' associated with f_{2r}.

3.12.4.1 Definition V admits a pseudounitary spin structure in a broad sense if and only if the structure group $U(p,q)$ is reducible to a subgroup of $O(Q'_{2p,2q})$ isomorphic to $(S_u)_e$, the enlarged pseudounitary spinoriality group associated with the $2r$-isotropic vector $y_1 \cdots y_{2r}$.

Such a definition is a generalization of the definitions given in 3.12.1, according to the proposition given in 3.12.3.2.

3.12.4.2 Proposition *V admits a pseudounitary spin structure in a broad sense if and only if there exists over V a $2r$-maximal totally isotropic subspace field such that with previous notation,*

$$\tilde{f}_{\beta'}(x) = \tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}(x),\ \tilde{f}_{\beta'}(x) = \mu_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x), \quad \mu_{\alpha'\beta'}(x) \in \mathbf{C}^*.$$

The demonstration is immediate (cf. exercises below).

3.12.5 Pseudounitary Conformal Spin Structures

3.12.5.1 Notation, Review, and Definitions

Let V be an almost pseudo-hermitian $2r$-dimensional manifold. $\xi(E, V, U(p,q), \pi)$ denotes the principal bundle of normalized orthogonal basis suitable for the pseudo-hermitian structure.[46]

We introduce naturally according to the already used Greub-extension of structures the following principal bundles: $\xi_{i_1} = \xi_1(A_1, V, CU_n(p,q), \pi_1)$, $P\xi_1 = \xi_{\theta_1 \circ i_1}(E'_1, V, PU(p+1, q+1), \tilde{\pi}_1)$. Then, we introduce a bundle $A(V)$ with typical fiber $\mathbf{C}^2$ provided with a quadratic hermitian form of signature $(1,1)$, the Whitney sum of two complex orthogonal line bundles, for this quadratic hermitian form $A(V) = l_0 \oplus l'_0$, and an "amplified" bundle of the tangent bundle $T_1(V) = T(V) \oplus A(V)$, i.e., $T_1(V) = \bigsqcup_{x \in V} (T_1)_x(V)$, where $T_1(x) = T_1(x) \oplus (l_0)_x \oplus (l'_0)_x$. The union of Clifford algebras $(Cl^{p+1,q+1})_x$ is naturally a vector bundle with typical fiber $Cl^{p+1,q+1}$, a bundle "locally trivial in algebra."

$U(p+1, q+1)$ acts as usual according to classical results[47] in the following way: for any $g \in RU(p+1, q+1)$, $w \in Cl^{p+1,q+1}$, $K_{\psi(g)}(w) = \pi(g)wg^{-1}$, where $\pi(g)wg^{-1}$ is dependent only on $\psi(g) \in U(p+1, q+1)$. $CU_n(p,q)$ acts naturally on such a bundle.

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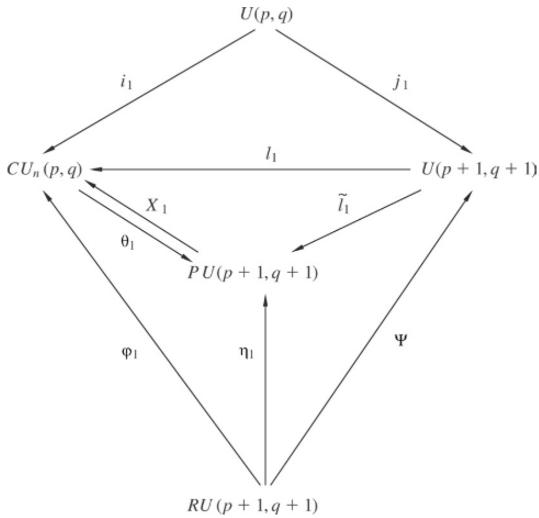
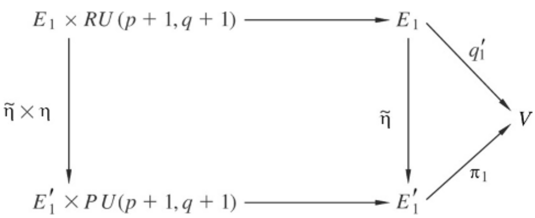
[46] Or more generally we can take for V an almost complex $2r$-dimensional manifold, which inherits an almost pseudo-hermitian structure according to previous remarks (3.1.1).

[47] Cf. for example, C. Chevalley, *The Algebraic Theory of Spinors*, op. cit.



It is known that $C_{2p,2q}$ can be obtained by linear combinations of $RO(2p, 2q)$. Then according to the definition of $Cl^{p,q}$, the same is true for $Cl^{p,q}$ relative to $RU(p, q)$. The mapping K_1 that sends $\varphi_1(g) \in CU_n(p, q)$—with $g \in RU(p+1, q+1)$—onto the morphism of $Cl^{p+1,q+1} : w \to \pi(g)wg^{-1}$ is well defined and constitutes a representation of $CU_n(p, q)$ into $Cl^{p+1,q+1} : K_{1\varphi(g)}w = \pi(g)wg^{-1}$. Thus we obtain a bundle denoted by $\mathrm{Clif}_1(V)$ and $CU_n(p, q)$ isomorphic to $PU(p+1, q+1)$ acts on such a bundle. This bundle is the analogous to the standard Clifford bundle.

3.12.5.2 Definition V admits a pseudounitary conformal spin structure in a strict sense if there exists a principal bundle $S_1 = (E_1, V, RU(p+1, q+1), q_1')$ and a morphism of principal bundles $\tilde{\eta}_1$ such that S_1 is a "4-fold covering," or rather a double two-fold covering since $E_N^2 = 1$, of $P\xi_1$ (or a 4-fold lifting of $P\xi_1$) with the following commutative diagram,



where the horizontal mappings correspond to right translations. S_1 is called the bundle of conformal spinor frames on V. The bundle of conformal spinors is the complex-vector-associated bundle of dimension 2^{2r+1} with typical fiber $\mathbf{C}^{2^{2r+1}}$:

$$\sigma = (E_1 \times_{RU(p+1,q+1)} \mathbf{C}^{2^{2r+1}}, V, RU(p+1, q+1), \mathbf{C}^{2^{2r+1}}).$$

We can now take again the previous proofs given in Section 2.6.2. We need to take care of the following changes: r becomes $2r$, e_N becomes E_N, ε_2 is now $(-1)^{2r+1}$, i.e., $\varepsilon_2 = -1$ (r even or odd), $\varepsilon = 1$ if $r - p$ is even, and $\varepsilon = -1$ if $r - p$ is odd. The f_r—r isotropic vector becomes f_{2r}.

We can now give the following results:

3.12.5.2.1 Proposition: Necessary and Sufficient Conditions for the Existence of a Pseudounitary Conformal Spin Structure in a Strict Sense

There exists on V a pseudounitary conformal spin structure in a strict sense if and only if:

(i) *There exists on V modulo the factor $\varepsilon_2 = -1$ an isotropic $(2r + 1)$-vector pseudo-field pseudo–cross section in the bundle $\mathrm{Clif}'_1(V)$.*

(ii) *The structure group of the principal bundle $P\xi_1$ is reducible, in the complexification, to a subgroup isomorphic to (S_{uc}) the pseudounitary conformal group in a strict sense associated with the $2r$-isotropic vector $y_1 \cdots y_{2r}$.*

(iii) *The complexified bundle $(P\xi_1)_{\mathbf{C}}$ admits local cross section and trivializing open sets with transition functions $\eta_1(g_{\alpha'\beta'}) \in RU(p+1, q+1)$ such that if $x \in U_{\alpha'} \cap U_{\beta'} \to \tilde{f}_{\alpha'}(x)$ defines locally the previous $(2r + 1)$-isotropic pseudofield, then*

$$\tilde{f}_{\beta'} = \tilde{g}_{\alpha'\beta'}(x)\tilde{f}_{\alpha'}(x)\tilde{g}^{-1}_{\alpha'\beta'}, \quad \textit{modulo } \varepsilon_2 = -1,$$

$$f_{\beta'} = (E_N)^2 N(\tilde{g}_{\alpha'\beta'}(x))\tilde{f}_{\alpha'}(x), \quad \textit{modulo } \varepsilon_2 = -1,$$

and where $(E_N)^2 = 1$ and therefore $\mathcal{A}_1 \simeq Z_2 \times Z_2$.

The obstruction class of the existence of a pseudounitary conformal spin structure on V : $K(V, \lambda)$, where $\lambda = l_1 \circ j_1 = i_1$, according to the commutative diagram shown in Figure 3.3 (with notation of 3.9.3.5 and of 3.11.4.5.3 and 3.11.4.5.5) is

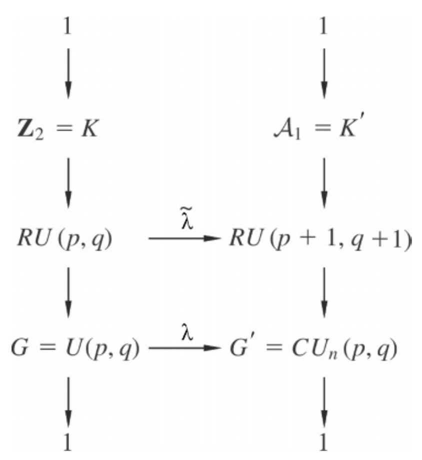


Fig. 3.3.

such that $K(V, \lambda) = \tilde{\lambda}_* K(P)$, *where* $\tilde{\lambda}_* : \check{H}(V, \mathbf{Z}_2) \to \check{H}(V, \mathbf{Z}_2 \times \mathbf{Z}_2)$ *denotes the homomorphism induced by the homomorphism* $\tilde{\lambda} : K \to K'$, *and* λ *is the natural inclusion of* $RU(p, q)$ *into* $RU(p+1, q+1)$.

It is sufficient to consider the result of 3.11.4.5.6.

3.12.5.2.2 Definition V admits a pseudounitary conformal spin structure in a broad sense if and only if the structure group $PU(p+1, q+1)$ of the principal bundle $P\xi_1$ is reducible, in the complexification, to a subgroup isomorphic to $(S_{uc})_e$, the enlarged pseudounitary conformal spinoriality group associated with the isotropic $2r$-vector $y_1 \cdots y_{2r}$.

Such a definition is a generalization of Definition 3.12.5.2.

3.12.5.2.3 Proposition *V admits a pseudounitary conformal spin structure in a broad sense if and only if there exists over V a* $(2r+1)$ *isotropic maximal totally isotropic field such that with the same notation as above,*

$$\tilde{f}_{\beta'} = \tilde{g}_{\alpha'\beta'}(x) \tilde{f}_{\alpha'}(x) \tilde{g}^{-1}_{\alpha'\beta'}, \qquad \text{modulo } \varepsilon_2 = -1,$$

$\tilde{g}_{\alpha'\beta'}(x) \in RU(p+1, q+1)$, $\tilde{f}_{\beta'} = \mu_{\alpha'\beta'}(x) \tilde{f}_{\alpha'}$, $\mu_{\alpha'\beta'}(x) \in \mathbf{C}^*$.

The proof is immediate.

3.12.6 Links between Pseudounitary Spin Structures and Pseudounitary Conformal Spin Structures

We choose the definition of the pseudounitary spin structure given in 3.12.1.1. A parallel study not given here can be done with Definition 3.12.1.2.

As in Section 2.7, using the previous necessary and sufficient conditions, we obtain the following results.

3.12.6.1 Proposition *If there exists an $RU(p, q)$-spin structure on V (i.e., a pseudounitary spin structure in a strict sense), then there exists an $RU(p+1, q+1)$-spin structure on V, the bundle of frames of the amplified tangent bundle.*

If there exists an $RU(p+1, q+1)$-spin structure on the bundle of frames of the amplified tangent bundle, then there exists a pseudounitary conformal spin structure in a strict sense over V.

If there exists a pseudounitary conformal spin structure in a strict sense over V (briefly speaking, a $CU_n(p, q)$-spin structure), since n is even if $r - p$ is odd, then there exists a pseudounitary spin structure over V.

The proof is immediate.

For the last point, since $\varepsilon_2 = -1$ (r even or r odd) and, taking into account that if $r - p$ is odd, then $\varepsilon = -1$, and noticing that $E_n^2 = 1$, we can deduce the existence of a $(2r)$-isotropic pseudo-vector field that satisfies the required conditions using previous necessary and sufficient conditions.

3.12.7 Concluding Remarks

The previous study show that pseudounitary spin structures and pseudounitary conformal spin structures appear as particular cases of standard pseudoorthogonal spin structures and conformal pseudoorthogonal ones.

What about symplectic spin structures and conformal symplectic spin structures? There exists a natural injection from the standard real symplectic group into the special pseudounitary group of type (n, n), i.e., $\rho_1 : Sp(2n, \mathbf{R}) \to SU(n, n)$ given in Satake.[48] Thus, one can develop canonically with algebraic and geometrical materials coming from Chapter 3 the construction and the study of symplectic spin structures and furthermore, of real conformal symplectic ones via the previous study.

3.13 Appendix

3.13.1 A Review of Algebraic Topology

The following books of reference contain all the necessary material: Séminaire Henri Cartan, 1967, W. A. Benjamin, New York, Topologie Algebrique, Espaces fibrés et homotopie, Cohomologie des groupes, suite spectrale, faisceaux, Roger Godement, Theorie des faisceaux, Actualités scientifiques et industrielles, 1252, Hermann, Paris 1964, Chapiter 1, Part 2, Generalites sur les complexes.

We want to give only some complements necessary for the understanding of the results of Greub and Petry recalled in 3.11.3. These classical results can be found, for example in James Dugundji, Topology, Allyn and Bacon, Boston, 1968, pp. 171–173, or in M. Zisman, Topologie Algébrique élémentaire, Librarie Armand Colin, Paris, 1972, pp. 223–227. We follow the method of Dugundji.

3.13.1.1 Classical definitions

Let E be any nonempty set. An n-simplex Σ_n in E is any set of $n+1$ distinct elements of E, namely $\Sigma_n = \{a_0, \ldots, a_n\}$. $a_0, \ldots, a_n$ are called the vertices of Σ_n; any $\Sigma_p \subset \Sigma_n$ is said to be a p-face of Σ_n.

An abstract simplicial complex K over E is a set of simplexes in E such that any face of a $\Sigma \in K$ is in K. We can associate a topological space with any simplicial complex. Let $A_0, \ldots, A_n$ be $(n+1)$ independent points in an affine space. The open geometric n-simplex Σ_n spanned by $A_0, \ldots, A_n$ is the set: $\{\sum_{i=0}^{n} \lambda_i A_i \mid \sum_{i=0}^{n} \lambda_i = 1,\ 0 < \lambda_i,\ i = 0, \ldots, n\}$ and will be denoted by $(A_0, \ldots, A_n)$.

Σ_n is the interior of the convex hull of $\{A_0, \ldots, A_n\}$ in the n-dimensional Euclidean space that these vertices span. A classical example is (A_0, A_1, A_2) that is a triangle without its boundary. The coefficients λ_i are called the barycentric coordinates of $M = \sum_0^n \lambda_i A_i$.

[48] I. Satake, *Algebraic Structures of Symmetric Domains*, Iwanami Shoten Publishers and Princeton University Press, 1980, p. 77.

3.13.1.2 Nerve of a covering

The mathematical process that associates with any open covering of a topological space a complex called its nerve is powerful since it permits to relate topological properties of a space to its algebraic ones.

3.13.1.2.1 Definition Let $\{U_a,\ a \in E\}$ be any open covering of a space. Let N be the complex over E defined by the following condition: $(a_0, \ldots, a_n)$ is a simplex of N if and only if $U_{a_0} \cap \ldots \cap U_{a_n} \neq \varnothing$. N is a complex called the nerve of $\{U_a,\ a \in E\}$.

3.13.2 Complex Operators and Complex Structures Pseudo-Adapted to a Symplectic Form

We want to develop the contents of section 3.1.2 and to review the classical results found by C. Ehresmann in the references given there. We follow the remarkable book of P. Liebermann and C. M. Marle.

3.13.2.1 Recalls on complex operators

Let W be a complex space of finite dimension n, and let $F = {}_{\mathbf{R}}W$ be the $2n$-dimensional real vector space obtained from W by restriction of scalars to $\mathbf{R}$. When considered as acting on F the multiplication by i, square root of -1, is a real linear operator J such that $J^2 = -Id_F$. Conversely, let V be a real space of finite dimension endowed with a real linear operator J such that $J^2 = -Id_V$. Then, necessarily, J is bijective and its inverse is $-J$. Since we have that $(\det J)^2 = \det(-Id_V) = (-1)^{\dim V}$, so V is of even dimension $2n$.

J is called a complex operator (or transfer operator) since it determines on V a structure of complex space. For any $\lambda = a + ib$, $a, b \in \mathbf{R}$ and $x \in V$ we define $\lambda x = ax + bJx$.

Briefly, we denote by (V, J) the space V endowed with a complex structure by means of the complex operator J. The complex dimension of V is n.

A classical example is the following:

Let $\{e_i\}$, $i = 1, \ldots, 2n$ be the canonical basis of $\mathbf{R}^{2n}$, and let J_0 be the operator defined by $J_0(e_k) = e_{n+k}$, $J_0(e_{n+k}) = -e_k$ for $1 \leq k \leq n$. Then J_0 is a complex operator that allows us to identify $\mathbf{R}^{2n}$ with $\mathbf{C}^n$ in the following way: to any $x = (x^1, \ldots, x^{2n})$, we associate $z = (x_1 + ix^{n+1}, \ldots, x^n + ix^{2n})$. In the same way, for any real $2n$-dimensional vector space V endowed with a basis $(\epsilon_1, \ldots, \epsilon_{2n})$ we can associate the complex operator defined by $J(\epsilon_k) = \epsilon_{n+k}$, $J(\epsilon_{n+k}) = -\epsilon_k$ for $1 \leq k \leq n$.

Let (V, J) be a complex structure on V with the complex operator J. A subspace F of V is said to be complex if F is conserved by J, that is if $JF = F$. Then it is a complex subspace of (V, J). A subspace F of V is said to be real with respect to the complex structure (V, J) if $F \cap JF = \{0\}$.

In particular, if $\dim F = n$, the real subspace F is said to be the real form of (V, J), since $V = F \oplus JF$. V can be then identified with the complexification of F.

3.13.2.2 Complements for pseudo-hermitian forms

We use the same notation and definitions as in 3.1.2.

3.13.2.2.1 Fundamental formulas

Let η be a pseudo-hermitian form on the complex finite-dimensional space W. We put $G(x, y) = Re(\eta(x, y))$, $\Omega(x, y) = -Im(\eta(x, y))$, where G is a real bilinear symmetric form and Ω a real skew–symmetric form.

For any real vector space U and for any element of its dual $\alpha \in U^*$, we will denote by $<\alpha, x>$ the value of α on $x \in U$. Let $\Omega^\flat$ be the classical isomorphism from ${}_{\mathbf{R}}W$ onto its real dual ${}_{\mathbf{R}}W^*$ defined by

$$\Omega(x, y) = -<\Omega^\flat(x), y>.$$

$\Omega^\sharp$ will denote the inverse of $\Omega^\flat$. Let $G^\flat$ be the isomorphism from ${}_{\mathbf{R}}W$ onto ${}_{\mathbf{R}}W^*$ defined by $< G^\flat(x), y >= G(x, y)$. We can verify the following relations, where J denotes the complex operator of V. For any $x, y \in W$, we have:

$$\begin{aligned} G(Jx, Jy) &= \ \ G(x, y) \\ \Omega(Jx, Jy) &= \ \ \Omega(x, y) \\ G(x, Jy) &= -\Omega(x, y) \\ \Omega(x, Jy) &= \ \ G(x, y) \end{aligned} \qquad \text{and } J = \Omega^\sharp \circ G^\flat. \qquad (A)$$

As an example, the standard hermitian form η_0 on $\mathbf{C}^n$ is defined by $\eta_0(z, z') = \sum_{k=1}^n z^k \bar{z}'^k$, for z, z' in $\mathbf{C}^n$. If we put $z = (x^1 + ix^{n+1}, \ldots, x^n + ix^{2n})$, then we have

$$\eta_0(z, z') = G_0(z, z') - i\Omega_0(z, z')$$

with $G_0(z, z') = \sum_{j=1}^{2n} x^j x'^j$ and $\Omega_0(z, z') = \sum_{k=1}^n \left(x^k x'^{n+k} - x'^k x^{n+k}\right)$.

3.13.2.2.2 Pseudo-adapted complex structures

Conversely we will study the following problem. Given a symplectic real space (V, Ω), does there exist a complex structure on V and a pseudo-hermitian form η such that $\Omega = -Im\,\eta$? We know that we can choose a symplectic basis on V that allows us to identify (V, Ω) with $(\mathbf{R}^{2n}, \Omega_0)$ and that J_0 is adapted to Ω_0. Whence, we can deduce that any symplectic real space (V, Ω) admits a complex operator adapted to Ω.

3.13.2.2.3 Proposition *(C. Ehresmann) Let (V, Ω) be a real symplectic space and let J be a complex operator on V. J is pseudo-adapted to Ω if and only if J is a symplectic automorphism, that is the following condition holds: $\Omega(Jx, Jy) = \Omega(x, y)$, for any x, y in V. If such a condition is satisfied, then the form η defined for any x, y in V by $\eta(x, y) = G(x, y) - i\Omega(x, y)$ with $G(x, y) = \Omega(x, Jy)$ is a pseudo-hermitian form on (V, J), the unique pseudo-hermitian form such that $\Omega = -Im\,\eta$.*

Proof. If J is pseudo-adapted, the properties stated in the proposition follow from the properties (A) given in 3.13.2.2.1. Conversely if $\Omega(Jx, Jy) = \Omega(x, y)$, then the real bilinear form G defined by $G(x, y) = \Omega(x, Jy)$ is symmetric since $G(y, x) = \Omega(y, Jx) = -\Omega(Jx, y) = \Omega(x, Jy) = G(x, y)$ and also nondegenerate. Then, the form η defined by $\eta(x, y) = G(x, y) - \mathrm{i}\Omega(x, y)$ is a nondegenerate pseudo-hermitian form on (V, J). The uniqueness of η is immediate.

3.13.2.2.4 Proposition *Let V be a real vector space of dimension $2n$ endowed with a symplectic form Ω and with a nondegenerate bilinear symmetric form G. Let K denote the automorphism $\Omega^\sharp \circ G^\flat$ of V, that is the automorphism K such that for any x, y in V we have: $G(x, y) = \Omega(x, Ky)$. Then the three following conditions are equivalent to each other:*

(i) *K is a complex operator, i.e., $K^2 = -Id_V$.*
(ii) $\Omega(Kx, Ky) = -\Omega(x, y)$, $\forall x, y \in V$.
(iii) $G(Kx, Ky) = G(x, y)$, $\forall x, y \in V$.

Proof. We have $\Omega(x, K^2y) = G(x, Ky) = G(Ky, x) = -\Omega(Kx, Ky)$. Since the form Ω is nondegenerate, this relation proves the equivalence between (i) and (ii). Moreover, we notice that

$$G(Kx, Ky) = \Omega(Kx, K^2y) = -\Omega(K^2y, Kx) = -G(x, K^2y)$$

which explains the equivalence of (i) and (iii).

3.13.2.2.5 Corollary—Definition. *(C. Ehresmann) If the operator K defined in 3.13.2.2.4 satisfies one of the equivalent conditions (i)–(iii), then it is pseudo-adapted to Ω The forms Ω and G are called interchanging forms by C. Ehresmann.*

3.13.2.2.6 Theorem *With the notations of 3.13.2.2.3 let J be a complex structure pseudo-adapted to Ω. An automorphism of V commutes with J and conserves the form η if and only if it conserves both of the forms Ω and G. The unitary group $U(V, \eta)$ coincides with the intersection of $Sp(V, \Omega)$ and $O(V, G)$:*

$$U(V, \eta) = Sp(V, \Omega) \cap O(V, G).$$

Moreover, if u is an automorphism of V commuting with J and preserves one of the forms Ω or G, then it preserves η and the other form.

Proof. First we want to prove that if u belongs to $U(V, \eta)$, then u belongs to $Sp(V, \Omega) \cap O(V, G)$ and that $u \circ J = J \circ u$. By definition we have

$$\begin{aligned} \eta(u(x), u(y)) &= G(u(x), u(y)) - i\Omega(u(x), u(y)) \\ &= \eta(u(x), u(y)) = G(x, y) - i\Omega(x, y). \end{aligned}$$

Comparing real and imaginary parts we find that $u \in Sp(V, \Omega) \cap O(V, G)$. Conversely, let us now suppose that u is an automorphism of V that conserves both G and Ω. We will show that u commutes with J. For any $x, y \in V$ we have $G(u(x), u(y)) = G(x, y)$. But $G(u(x), u(y)) = \Omega(u(x), J(u(y)))$ by definition. On the other hand we have that $G(x, y) = \Omega(x, J(y)) = \Omega(u(x), J(u(y)))$ since $u \in Sp(V, \Omega)$. So we find that $\Omega(u(x), J(u(y))) = \Omega(u(x), u(J(y)))$ for all $x, y \in V$. Since Ω is nondegenerate it follows that $J \circ u = u \circ J$. It is then evident that $u \in U(\eta, V)$.

To prove the last statement of the theorem, let us assume that u is an automorphism of V commuting with J, and that u preserves, say, Ω (that is $u \in Sp(V, \Omega)$). Then for all $x, y \in V$ we have $G(x, y) = \Omega(x, Jy) = \Omega(u(x), u(J(y))) = \Omega(u(x), J(u(y))) = G(u(x), u(y))$. Thus $u \in O(V, \eta)$ and therefore u preserves η.

3.13.3 Some Comments about Spinoriality Groups

We want to give some comments about A. Crumeyrolle's classical spinoriality groups (see for example Crumeyrolle A., Groupes de spinorialité, Ann. Inst. Henri Poincaré, Sect. A, vol. XIV, n° 4, pp. 309–323, 1971, and also 3.10 above). The reader will find all the required information concerning bilinear forms in Chapter I of Chevalley's book: The algebraic theory of spinors, op. cit., and in N. Bourbaki, XXIV, Formes sesquilineares et formes quadratiques, op. cit.

3.13.3.1 Some Recalls

First, we recall the following result (see C. Chevalley, op. cit., 2.7 pp. 60–61). Let M be a finite-dimensional vector space over a field K and let Q be a quadratic form on M, where the associated bilinear form is nondegenerate. Let K' be an overfield of K, let M' be the space over K' obtained from M by extending to K' the basic field and let Q' be the quadratic form on M' that extends Q. Then the Clifford algebra C' of Q' may be identified with the algebra deduced from the algebra C of Q by extending the basic field to K.

From now on we will assume that E is a vector space over $\mathbf{R}$ or $\mathbf{C}$, and that B is a nondegenerate symmetric bilinear form on E.

Two vectors $x, y \in E$ are called orthogonal to each other if $B(x, y) = 0$. Let P is a subspace of E. Then $P^{\perp}$ is the subspace consisting of all vectors orthogonal P. Two subspace P_1 and P_2 are called orthogonal if $P_1 \subset P_2^{\perp}$ or, equivalently, if $P_2 \subset P_1^{\perp}$. P is called isotropic if $P \cap P^{\perp} \neq \{0\}$. A nonzero subspace P is called totally isotropic if it is orthogonal to itself or, equivalently, if the restriction of B to P is identically zero. A totally isotropic subspace P is called maximal if for any other totally isotropic subspace P_1, $P \subset P_1$ implies $P = P_1$.

3.13.3.2 Proposition (C. Chevalley, op. cit. I.4.3) *All maximal totally isotropic of E have the same dimension r. This common dimension r is called the Witt index of B and satisfies the relation $2r \leq \dim(E)$. The orthogonal group of (E, B) acts transitively on the set of maximal isotropic subspaces.*

3.13.3.3 Witt Decomposition

With the assumptions and notation as above, a Witt decomposition of E is a decomposition in a direct sum (not an orthogonal sum) of E into three subspaces

$E = F \oplus F' \oplus G$, where F and F' are totally isotropic, while G is nonisotropic and orthogonal to both F and F'.

Over the field $\mathbf{C}$, if the dimension of the space E is $n = 2r$, all maximally isotropic subspaces have dimension r. In this case there exists Witt decomposition $E = N + P$ into a sum of two totally isotropic subspaces. Moreover one can form a special basis of E consisting of vectors $x_i \in N$, $y_i \in P$, $i = 1, 2, \dots r$, such that $2B(x_i, y_j) = \delta_{i,j}$. Such a basis is called Witt basis. Notice that automatically $B(x_i, x_j) = B(y_i, y_j) = 0$.

3.13.3.4 Special Witt Bases

Starting from the study of pure spinors introduced first by Elie Cartan and then studied algebraically by Claude Chevalley in Chapter III of his remarkable book. The algebraic theory of spinors (op. cit) A. Crumeyrolle introduced special Witt bases he called "real Witt bases." In many papers such as "Groupes de spinoriality," Ann. Inst. H. Poincaré, vol. XIV, 4, 1971, pp. 309–323, "Derivations, formes et operateurs usuels sur les champs spinoriels des varietes differentiables de dimension paire," Ann. Inst. H. Poincaré, vol A 16 no 3, 1972, pp. 171–201, "Spin fibrations over manifold and generalized twistors," Proc. Symp. Pure Math., vol 27, 1975, pp. 53–67, A. Crumeyrolle defined and studied classical spinoriality groups.

Let (E, B) be a real $n = 2l$-dimensional vector space E equipped with a nondegenerate symmetric bilinear form B of signature (p, q), $p \neq q$ and let (E', B') be its complexification. We will denote by $^{-}$ the complex conjugation in $E' = E + iE$. It is known that the Witt index of (E, B) is $r = \min(p, q)$. Then $E' = F + Z$, where F and Z are two maximal totally isotropic subspaces F and Z, (both of complex dimension l, and there exists a special Witt basis $\{x_\lambda, y_\lambda, x_i, y_i,\ 1 \leq \lambda \leq r,\ 1 \leq i \leq l - r\}$ of E' with the following properties:

(i) $x_\lambda, x_i \in F$,
(ii) $y_\lambda, y_i \in Z$,
(iii) $\overline{x_\lambda} = x_\lambda$, $\overline{y_\lambda} = y_\lambda$, $\overline{y_i} = \delta y_i$, where $\delta = 1$ if $p > q$, and $\delta = -1$ if $p < q$.

If we write $Q(x) = (x^1)^2 + \cdots + (x^p)^2 - (x^{p+1})^2 - \cdots - (x^{p+q})^2$, $n = p + q = 2r$, $p < q$, then we can take:

$$F = Vect\{\frac{e_1 + e_n}{\sqrt{2}}, \frac{e_2 + e_{n-1}}{\sqrt{2}}, \dots, \frac{e_p + e_{q+1}}{\sqrt{2}}, \frac{ie_{p+1} + e_q}{\sqrt{2}}, \dots, \frac{ie_r + e_{n-r+1}}{\sqrt{2}}\},$$

$$Z = Vect\{\frac{e_1 - e_n}{\sqrt{2}}, \frac{e_2 - e_{n-1}}{\sqrt{2}}, \dots, \frac{e_p - e_{q+1}}{\sqrt{2}}, \frac{ie_{p+1} - e_q}{\sqrt{2}}, \dots, \frac{ie_r - e_{n-r+1}}{\sqrt{2}}\}.$$

3.13.3.5 Example

The use of these "real" special Witt bases is indispensable for determination of the results found by A. Crumeyrolle, such as matrix expressions.

As an example we give here the hints of the proof of Proposition 3.10.2.

We set $f = y_1 \dots y_r$ with $y_j = \frac{ie_j - e_{n-j+1}}{\sqrt{2}}$, where the $\{e_j\}$ constitute an orthonormal basis of E. We set $x_j = \frac{ie_j + e_{n-j+1}}{\sqrt{2}}$. Classically, by using τ, the main antiautomorphism of the Clifford algebra and Witt theorem (cf. below) the condition $\gamma f = \pm f$ is equivalent to $\gamma f \gamma^{-1} = f$. We set $y_{j'} = \gamma y_j \gamma^{-1}$ and then $y_{i'} = A^j_{i'} y_j$ and therefore $x_{i'} = \bar{A}^j_{i'} x_j$ where $x_{j'} = \gamma x_j \gamma^{-1}$, with $\gamma \in RO(Q)$. Since the Witt basis $\{x_i, y_j\}$ is applied onto the Witt basis $\{x_{i'}, y_{j'}\}$ we get: $\sum_j A^j_{i'} \bar{A}^j_{k'} = \delta_{i'k'}$. If we consider $y_1 y_2 \dots y_r$ as an element of $\bigwedge^r F'$, with previous notations, we find immediately the results given in Proposition 3.10.2.

Finally, we recall the classical Witt theorem (see for example C. Chevalley, op.cit., p. 16, I.4.1). Let (E, Q) be a quadratic regular finite-dimensional space over K. Every Q-isomorphism of a subspace N of E with a subspace M may be extended to an operation of the group $O(Q)$.

3.14 Exercises

(I) Proof Theorem 3.5.2.3

(1) First show that the group $PU(F)$ of Definition 3.4.1 is included in $CU_n(p, q)$, following the results of the Proposition 3.4.4.2.

(2) Prove the converse by using the construction of covering groups for $CU_n(p, q)$ made in Section 3.9.2. Use results of Section 2.4.2.

(II)

(1) Determine the real associative algebras $Cl^{0,1}, Cl^{1,0}, Cl^{1,1}, Cl^{2,2}, Cl^{3,3}$.

(2) Using the classical periodicity of the real algebras $C^+_{2p,2q}$ study the periodicity of the algebras $Cl^{p,q}$.

(III)

Let $\mathbf{R}^n, n = 2r$ be endowed with a negative definite quadratic form φ. Let $\{e_1, \dots, e_n\}$ be an orthonormal base of $\mathbf{R}^n$. We assume that $\mathbf{R}^n$ is provided with a linear orthogonal transformation such that $J^2 = -\mathrm{Id}$. $Cl(n)$ denotes the Clifford algebra of $(\mathbf{R}^n, \varphi)$.

(1) One defines an endomorphism of $\mathbf{R}^n$ by $J_\theta = \cos\theta \cdot I + \sin\theta J$, where I denotes the identity mapping of $\mathbf{R}^n$ and $\theta \in \mathbf{R}$.

(a) Show that for any $x \in \mathbf{R}^n$, x and $J(x)$ are orthogonal.

(b) Show that from J_θ one can deduce an automorphism of the algebra $Cl(n)$.

(c) We set

$$D_0(v_1 v_2 \cdots v_q) = \sum_{j=1} (v_1 \cdots J v_j \cdots v_q),$$

where $v_1, \dots, v_q \in Cl(n)$. Show that D_0 is a derivation of the algebra $Cl(n)$.

(d) Show that there exists an orthonormal basis of $\mathbf{R}^n$ of the type $(e_1, Je_1, e_2, Je_2, \ldots, e_r, Je_r)$.

(2) We set $Cl'(n) = Cl(n) \otimes_{\mathbf{R}} \mathbf{C}$. We will assume that $Cl'(n)$ is the Clifford algebra constructed from $\mathbf{C}^n$ endowed with the quadratic form Q' that extends Q naturally by complexification. We set $\varepsilon_k = \frac{1}{2}(e_k - iJe_k)$, $\bar{\varepsilon}_k = \frac{1}{2}(e_k + iJe_k)$, $k = 1, 2, \ldots, r$. We will admit that we obtain, then, a basis of $\mathbf{C}^n$.

(a) Write the multiplicative table of this basis in the Clifford algebra $Cl'(n)$.

(b) We set

$$\mathcal{L}(\varphi) = -\sum_{k=1}^{r} \varepsilon_k \varphi \bar{\varepsilon}_k, \ \bar{\mathcal{L}}(\varphi) = -\sum_{k=1}^{r} \bar{\varepsilon}_k \varphi \varepsilon_k, \text{ for any } \varphi \in Cl'(n).$$

Show that $\mathcal{L}$ and $\bar{\mathcal{L}}$ are well defined independently of any orthonormal basis such that $\{e_1, Je_1, \ldots, e_r, Je_r\}$.

(c) We set $\mathcal{H} = [\mathcal{L}, \bar{\mathcal{L}}]$. Write the multiplicative table for the bracket [,] of the elements $\mathcal{L}, \bar{\mathcal{L}}, \mathcal{H}$.

(3) Show that $\mathcal{H}(\varphi) = w\varphi + \varphi w - r\varphi = \bar{w}\varphi - \varphi\bar{w} + r\varphi$ if $w = -\sum \varepsilon_k \bar{\varepsilon}_k$, $\bar{w} = -\sum \varepsilon_k$ and that $\mathcal{H}(\varphi) = \sigma\varphi + \varphi\sigma$ if $\sigma = \frac{1}{2i} \sum_k e_k \cdot Je_k$, for any $\varphi \in Cl'(n)$.

(IV) Prove Proposition 3.9.2.2 and Lemma 3.9.2.3.1. Hints: For Proposition 3.9.2.2 follow the same method as in Section 2.4.2.2. For Lemma 3.9.2.3.1 use the fact that the space contains a hyperbolic plane (cf. E. Artin, Algèbre géometrique, op.cit., p. 126).

(V) Prove Lemmas (1), (2), (3) given in 3.9.3 and Corollary 3.9.3.4 (cf. also 2.5.1.2.1).

(VI) Prove 3.9.3.5.2 explicitly. Hints: Use the same method as in Section 2.5.1.2.

(VII)

(1) Prove the propositions given in 3.10.1.2, 3.10.1.3, 3.10.1.4, and 3.10.1.5 (see Appendix 3.13.3).

(2) Study the special case of the Minkowski space with Q of signature (1, 3).

(3) Prove the results given in 3.10.2.2 concerning the enlarged spinoriality group associated with f_{2r} by using the same method that was used in Section 2.5.1.4.

(4) Show the characterization given in 3.10.3.5 using the same method that was used in Section 2.5.1.4 (see also Appendix 3.13).

(VIII) Prove the necessary and sufficient conditions given in 3.11.4.2.

(IX) Prove in detail Proposition 3.12.2.1, 3.12.3.1, 3.12.3.2, 3.12.4.2.

(X) Construction of Dirac operators[49]

Let $Cl(E)$ denote the Clifford algebra of (E, q), a quadratic regular standard n-dimensional Euclidean space, and let M denotes an n-dimensional riemannian manifold. $Cl(M)$ denotes the corresponding bundle, the fiber of which at $m \in M$ is the Clifford algebra $Cl(T(M)_m)$ of the tangent space where the quadratic form is q_m, or technically the opposite form $(-q_m)$.

Let φ be the natural linear isomorphism from the exterior algebra $\wedge E$ onto $Cl(E)$ such that $\varphi(\wedge^p E) = Cl_p(E)$ is the subspace of p skew-symmetric elements of $Cl(E)$. The $\binom{n}{p}$ products e_J defined as $\{e_J = e_{i_1} e_{i_2} \cdots e_{i_p}, i_1 < i_2 \cdots < i_p\}$ constitute a basis of $Cl(E)$. Since we can identify, using the quadratic form q, E with its dual space E^*, we are led to identify $T(M)$ with $T^*(M)$, $\wedge T(M)$ with $\wedge T^*(M)$, and to consider the natural linear isomorphism Φ, determined by φ from $\wedge T(M)$ or $\wedge T^*(M)$ onto $Cl(M)$.

The corresponding C^∞ cross sections of $Cl(M)$ are Cliffordian fields defined on M and constitute an algebra $\Gamma Cl(M) = C(M)$. The vector fields $X \in \Gamma T(M)$, and the exterior differential forms $w \in \mathcal{A}(M) = \Gamma \wedge T^*(M)$ can be considered, via Φ, as Cliffordian fields on M.

Let Pin E be in $Cl(E)$ the classical twofold covering of $O(E)$ and let Spin E be its connected component, included in $Cl_+(E)$, the universal covering of the group of rotations $SO(E)$ of E. $Cl_2(E)$ with the usual bracket of $Cl(E)$ is the Lie algebra of $O(E)$ and Pin E.

The bundle $Cl(M)$ contains bundles Pin M and Spin M with respective groups Pin E and Spin E.

(a) Show that if we set $q(u) = 2^{-n} Tr(l(u^\tau u))$ with notation of Chapter 1, q is a quadratic form defined on $Cl(E)$:

(b) Show that the $\{e_I\}$ constitute an orthonormal Euclidean basis for $(Cl(E), q)$ and that $q(u)$ is the component relative to the unity of $u^\tau u$ in the basis $\{e_I\}$.

(c) For any g in Pin(E), $q(g(u)) = q(u)$ and $q(ug) = q(u)$.

The Clifford algebra $Cl(E)$ is a semisimple algebra and inherits a minimal faithful module, unique up to isomorphism, called the space of spinors $S(E)$, where the groups Pin E and Spin E are represented naturally. $S(E)$ is isomorphic to any minimal faithful left ideal of $Cl(E)$ as a $Cl(E)$-left module.

(2) (a) Show that the interior automorphisms $u \to gug^{-1}$, with $g \in$ Spin(E), leave E and any $Cl_p(E)$ invariant globally. Thus, Spin E acts on E and by left translation on $Cl(E)$ or $S(E)$.

(b) Let μ be multiplication from $E \otimes Cl(E)$ into $Cl(E)$ or from $E \otimes S(E) \to S(E)$. Show that μ is a surjective morphism of Spin E modules.

Let $\tilde{P}(M)$ be the principal bundle, with group Pin E, a twofold covering of the principal bundle $P(M)$ of orthonormal basis of $T(M)$, and let $S(M)$ be the corresponding spinor bundle that determines on M the chosen spin structure.

[49] The method is due to R. Deheuvels, Quelques applications des algèbres de Clifford à la géométrie, *Riv. Mat. Univ. Parma*, 4, 14, 1988, pp. 55–67.

Let ∇ be the riemannian covariant differential

$$\nabla : \Gamma(\otimes T(M)) \to \Gamma T^*(M) \otimes \Gamma(\otimes T(M)).$$

Since the metric tensor is parallel ($\nabla g = 0$), by passing to the quotient we determine ∇, the covariant derivation of Cliffordian fields:

$$\nabla : \Gamma Cl(M) \to \Gamma T^*(M) \otimes \Gamma Cl(M) = \Gamma T(M) \otimes \Gamma Cl(M).$$

(3) (a) Show that the quadratic form q on $Cl(E)$ determines a scalar product $(u \mid v)$ between Cliffordian fields $u, v \in \Gamma Cl(M)$ with values in the set of functions C^∞ on M and that $\nabla(u \mid v) = (\nabla u \mid v) + (u \mid \nabla v)$.

(b) Show that the curvature form $R \in Cl_2(M) \otimes \wedge^2 T^*(M)$ is obtained via the lifting in the bundles of the mapping from $\Gamma \wedge^2 T(M)$ into the set of derivations of the algebra $\Gamma Cl(M)$:

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - [\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X}].$$

(c) Show that the riemannian connection determines a connection on the bundle $\tilde{P}(M)$ and then a covariant derivation ∇ on the bundle $S(M)$, which acts on the spinor fields, i.e.,

$$\nabla : \Gamma S(M) \to \Gamma T(M) \otimes \Gamma S(M).$$

The Dirac operator D is defined on $\Gamma Cl(M)$ and on $\Gamma S(M)$ as the composite of the covariant derivative and the multiplication μ, $\Gamma(T(M))$ considered as included in $\Gamma Cl(M)$:

$$\Gamma Cl(M) \to^{\nabla} \Gamma T(M) \otimes \Gamma Cl(M) \to^{\mu} \Gamma Cl(M),$$
$$\Gamma S(M) \to^{\nabla} \Gamma T(M) \otimes \Gamma S(M) \to^{\mu} \Gamma S(M).$$

(4) Locally a vector field X can be written $X = \sum e_i X^i$. Show that in an orthonormal basis, the Dirac operator $\tilde{\mathcal{D}}$ is equal to $\sum e_i \nabla e_i$, in $\Gamma Cl(M)$ and in $\Gamma S(M)$.

(5) Let $E = E_{r,s}$ be the standard pseudo-Euclidean space provided with an orthonormal basis $\{e_1, \dots, e_n\}$. We agree to identify e_j, $1 \leq j \leq n$, with its image in the Clifford associated algebra $C = C(E_{r,s})$. We recall that $q(x) \cdot 1_{\mathbf{C}} = (e_1 x^1 + e_2 x^2 + \cdots + e_n x^n)^2$ and that the Dirac operator can be expressed as $\tilde{\mathcal{D}} = e_1(\partial/\partial x^1) + e_2(\partial/\partial x^2) + \cdots + e_n(\partial/\partial x^n)$.

(a) Show that $\tilde{\mathcal{D}}$ is independent of the choice of the orthonormal basis of E. $\tilde{\mathcal{D}}$ acts on the spinor differentiable fields, differentiable functions of E with values in a space of spinors S of E, by derivation and "multiplication" by elements of C, and $\tilde{\mathcal{D}}$ acts also on the differentiable functions of $E_{r,s}$, with values in its Clifford algebra C. The square $\tilde{\mathcal{D}}^2$ is a scalar operator, or "diagonal": $\tilde{\mathcal{D}}^2 = \triangle$, which is the Laplacian of $E_{r,s}$.

(6) (a) Show that on the standard Euclidean space E_n, if $\triangle_{e_j} = \frac{\partial}{\partial x^j}$, then $\triangle_{e_j} e_k = 0$, $\forall\, j, k$ and $\tilde{\mathcal{D}}^2 = \sum_j e_j(\partial/\partial x^j) \cdot \sum_k (\partial/\partial x^k) = \sum(\partial^2/(\partial x^j)^2) = \triangle$. A spinor field s on $E_{r,s}$ such that $\tilde{\mathcal{D}} = 0$ is called harmonic.

(7) We put $n = 2$.

(a) Verify that $C(E_2) = \mathcal{M}(2, \mathbf{R})$, $S(E_2) = \mathbf{R}^2$.

(b) Show that E_2 can be embedded into $C(E_2)$ by

$$e_1 \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and } e_1 e_2 = J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with $J^2 = -I$.

(c) Verify that J can be identified with i if $\mathbf{R}^2$ is identified with the complex line $\mathbf{C}$. A spinor field is then a function s on E_2 with values in $\mathbf{R}^2$. Verify that

$$\tilde{\mathcal{D}}s = \left(e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y}\right) s = \begin{pmatrix} -\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix}.$$

$\tilde{\mathcal{D}}$ is then the classical Cauchy–Riemann operator and the harmonic spinors (which satisfy $\tilde{\mathcal{D}}s = 0$) are analytic functions $u + iv$ of the complex variable $x + iy$.

(d) Show that $\tilde{\mathcal{D}}$ acts on a vector field of $E_2 : X = ae_1 + be_2$ by

$$\tilde{\mathcal{D}}X = \begin{pmatrix} -\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} -a & b \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} & -(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}) \\ \frac{\partial b}{\partial x} - \frac{\partial a}{\partial y} & \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y} \end{pmatrix} = (\operatorname{div} X)I + (\operatorname{rot} X)J.$$

(8) We put $n = 3$.

(a) Verify that $C(E_3) = \mathcal{M}(2, \mathbf{C})$, $S(E_3) = \mathbf{C}^2$.

(b) Show that E_3 can be embedded into $C(E_3)$ via the Pauli matrices

$$e_1 \to \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \to \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 \to \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $J = e_1 e_2 e_3 = iI$.

$\tilde{\mathcal{D}}$ acts on a spinor field s, with complex components $u(x, y, z)$, $v(x, y, z)$ (s is a function on E_3 with values in $\mathbf{C}^2$) by

$$\tilde{\mathcal{D}}s = \begin{pmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & -\frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial z} \end{pmatrix}.$$

The harmonic spinors of E_3—which satisfy $\tilde{\mathcal{D}}s = 0$—are generalizations of analytic functions.

(9) We put $n = 4$ and we consider $E_{1,3}$.

(a) Show that $C(E_{1,3}) = \mathcal{M}(2, \mathbf{H})$ and that $C(E_{1,3}) \otimes \mathbf{C} = \mathcal{M}(4, \mathbf{C})$. We consider the complex Dirac operator that acts on the complex space $S = \mathbf{C}^4$.

A set of Dirac matrices is a set of four complex square matrices of order four, $\gamma_0, \gamma_1, \gamma_2, \gamma_3$, which satisfy the following relations:

(i) $(\gamma_0)^2 = 1$, $(\gamma_1)^2 = (\gamma_2)^2 = (\gamma_3)^2 = -1$;

(ii) $\gamma_i\gamma_j + \gamma_j\gamma_i = 0, \forall\, i \neq j$.

(b) Show that we can then define a Clifford mapping from $E_{1,3}$ into $\mathcal{M}(4, \mathbf{C})$ by the following choice:

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \gamma_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix},$$

where the σ_j are the classical Pauli matrices introduced in 8(b) above.

(c) Verify that

$$\tilde{\mathcal{D}} = \gamma_0 \frac{\partial}{\partial x^0} + \gamma_1 \frac{\partial}{\partial x^1} + \gamma_2 \frac{\partial}{\partial x^2} + \gamma_3 \frac{\partial}{\partial x^3}$$

and that

$$\tilde{\mathcal{D}}^2 = \Box = \frac{\partial^2}{(\partial x^0)^2} - \frac{\partial^2}{(\partial x^1)^2} - \frac{\partial^2}{(\partial x^2)^2} - \frac{\partial^2}{(\partial x^3)^2}.$$

(XI) Prove the results given in Section 3.11.4.5

(1) Proof of 3.11.4.5.2. Prove the following lemmas:

Lemma I *A G-principal bundle admits a Γ structure if and only if there are continuous maps $\gamma_{ij} : U_{ij} \to \Gamma$ such that*

(a) $\gamma_{ij}(x)\gamma_{jk}(x) = \gamma_{ik}(x)$, $x \in U_{ijk}$ *(with obvious notation),*

(b) $\rho \cdot \gamma_{ij} = g_{ij}$, where $g_{ij} : U_{ij} \to G$ are the transition functions assoicated with a simple covering (U_{ij}) of B by open sets and with a system of local cross sections.

Lemma II *A space is called an L-space if every open covering has a simple refinement. (We recall that an open covering $\{U_i\}$ of a topological space is called simple if all the nonempty intersections $U_{i_1} \cap \cdots \cap U_{i_p}$ are contractible.)*

Let $U = \{U_i\}$ be an open covering of B such that P is trivial over U_i. Since B is an L-space, we may assume that the covering is simple. Then the transition mappings $g_{ij} : U_{ij} \to G$ lift to continuous mappings $\gamma_{ij} : U_{ij} \to \Gamma$. Now consider a nonempty triple intersection U_{ijk} and set for $x \in U_{ijk} : p_{ijk}(x) = \gamma_{jk}(x)\gamma_{ik}(x)^{-1}\gamma_{ij}(x)$. Then $\rho p_{ijk}(x) = g_{jk}(x)g_{ik}(x)^{-1}g_{ij}(x) = e$, and so $p_{ijk}(x) \in K$, $x \in U_{ijk}$. Since U_{ijk} is connected and K discrete, the p_{ijk} must be constant, and then define a 2-cochain in the nerve $N(U)$ with values in the abelian group K. This cochain will be denoted by p, $p(i, j, k) = p_{ijk}(x)$, $x \in U_{ijk}$. Show that p is a cocycle.

Lemma III *Show that the cohomology class represented by the cocycle p is independent of the choice of the local section σ_i and the liftings γ_{ij}.*

(2) p determines an element of $\check{H}^2(N(U), K)$. Passing to the direct limit (over all simple coverings), we obtain an element $K(P, \rho) \in \check{H}^2(B, K)$, which is called the Γ-obstruction. Show that a principal bundle admits a Γ-structure if and only if the class $K(P, \rho)$ vanishes.

(3) Prove Proposition 3.11.4.5.7 (Greub and Petry, op. cit. pp. 240–242) for the case of $O(n)$ bundles.
Hints: Study the special case of $O(1)$ bundles. Then proceed by induction.

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